Why is the Fundamental Theorem of Calculus, Part I, True?

The following is an informal argument about why the Fundamental Theorem of Calculus, Part I, is true. It states:

Suppose $f$ is a continuous function and that $F$ is defined by $F(x) = \int_a^x f(t) \, dt$ (where $a$ is some constant). Then $F'(x) = f(x)$.

Why? To compute $F'(x)$, we go back to the definition of a derivative.

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{\int_a^{x+h} f(t) \, dt - \int_a^x f(t) \, dt}{h}.$$  

But the numerator $\int_a^{x+h} f(t) \, dt - \int_a^x f(t) \, dt = \int_x^{x+h} f(t) \, dt$ (because $\int_a^{x+h} f(t) \, dt = \int_a^x f(t) \, dt + \int_x^{x+h} f(t) \, dt$).

Therefore

$$F'(x) = \lim_{h \to 0} \frac{\int_x^{x+h} f(t) \, dt}{h}.$$  

Now look at the figure below:

As $h \to 0$, the area of the rectangle is always $hf(x)$.

As $h \to 0$, $f(x + h) \to f(x)$ (because $f$ is continuous) and therefore

Area under graph over $[x, x + h] = \int_x^{x+h} f(t) \, dt \to \text{area of rectangle} = hf(x)$.

So $F'(x) = \lim_{h \to 0} \frac{\int_x^{x+h} f(t) \, dt}{h} = \lim_{h \to 0} \frac{hf(x)}{h} = f(x).$  

(OVER)
Why is \( f \) required to be continuous?

If \( f \) is not continuous, the argument doesn't work: in the figure below, the region under the graph over \([x, x + h]\) doesn't come toward coinciding with the rectangle as \( h \to 0 \). Therefore replacing \( \int_x^{x+h} f(t) \, dt \) with \( hf(x) \), as we did above, isn't justified.