EXAM I

Math 109 / Music 109A, Spring 2014

Name ___________________________ Id __________

Each problem is worth 10 points.

1. **Aural**: Notate the rhythm (one measure each).

   (a) \[ \frac{4}{4} \]

   (b) \[ \frac{12}{8} \]

   Circle the triad type.

   (c) \[
   \begin{array}{c}
   \text{major} \\
   \text{minor} \\
   \text{diminished} \\
   \text{augmented}
   \end{array}
   \]

   (d) \[
   \begin{array}{c}
   \text{major} \\
   \text{minor} \\
   \text{diminished} \\
   \text{augmented}
   \end{array}
   \]

2. Sketch the graphs of these functions by starting with a more basic function and applying one or more geometric transformations (shifts or stretches). Use the space on page 4 if you need it.

   (a) \[ f(x) = -x^2 + 2 \]

   (b) \[ g(x) = -1 + \cos \frac{x}{4} \]

3. For the following pairs of integers \( m, n \), find the numbers \( q \) and \( r \) whose existence is asserted in the division algorithm \( n = qm + r \):

   (a) \[ 7, -22 \]; \[ -22 = -4 \cdot 7 + 6 \]

   \[ q = -4 \quad r = 6 \]

   (b) \[ 3, 102\ell + 4 \], where \( \ell \) some integer.

   \[ 102\ell + 4 = (34\ell + 1) \cdot 3 + 1 \]

   \[ q = 34\ell + 1 \quad r = 1 \]
4. Write the indicated note as a whole note, choosing and notating an appropriate clef.

(a) \[ \text{F}_4 \]  
(b) \[ \text{A}_5^\# \]  
(c) \[ \text{B}_2^\# \]

5. For the set \( \{(a, b) \in \mathbb{Z}^2 \mid b \neq 0 \} \) show that the relation \( \sim \) defined by \( (a, b) \sim (a', b') \) iff \( ab - a'b = 0 \) is an equivalence relation. Explain how the set of equivalence classes are in one-to-one correspondence with the set of rational numbers \( \mathbb{Q} \).

OR

For the set \( \mathbb{Z} \) and a fixed positive integer \( m \), show that the relation \( \equiv \) defined by \( k \equiv l \text{ iff } m \mid (k - l) \) is an equivalence relation. Explain why there are exactly \( m \) equivalence classes.

\[ (i) \quad a b - a'b = 0 \quad \text{so} \quad (a, b) \sim (a', b') \quad \text{for any pair.} \]

Hence reflexive

\[ (ii) \quad \text{Assume} \quad (a, b) \sim (a', b') \quad \text{Then} \quad a b' - a' b = 0 \quad \text{Multiply by} \quad -1 \quad \text{to get} \quad a b - a' b' = 0 \quad \text{Thus} \quad (b, b') \sim (a, b). \]

Hence symmetric

\[ (iii) \quad \text{Assume} \quad (b, b') \sim (a, a') \quad \text{and} \quad (a', b') \sim (c, c') \quad \text{Then} \quad a b' - a' b = 0 \quad \text{Multiply by (equation b' = b')} \quad \text{and} \quad a c' - a' c = 0 \quad \text{Multiply by (equation c' = b')} \quad \text{add} \quad (a b - a' b) + (c' b - c b') = 0 \]

\[ a b - a' b = 0 \quad \text{and} \quad a c' - a' c = 0 \]

\[ \text{By b and add:} \quad (a b - a' b) + (c' b - c b') = 0 \]

\[ a b - a' b = 0 \quad \text{and} \quad a c' - a' c = 0 \]

\[ \text{Hence Transitive.} \]

Associate the class \( [(a, b)] \) to \( \frac{a}{b} \in \mathbb{Q} \).

If \( (a, b) \sim (a', b') \) then \( a b - a' b = 0 \quad \text{so} \quad \frac{a}{b} = \frac{a'}{b'} \), so this association does not depend on the class representative.

This defines a function \( \frac{a}{b} \rightarrow \mathbb{Q} \) (where \( S = \{(a, b) \in \mathbb{Z}^2 \mid b \neq 0 \}) \).
(i) \( k \equiv k \mod m \), so \( m \mid (k-k) \). Hence \( k \equiv k \). Hence reflexive.

(ii) Assume \( k \equiv l \). Then \( m \mid (k-l) \), so \( k-l = qm \) for some \( q \in \mathbb{Z} \).

Multiplying by \(-1\) yields \( l-k = -qm \), so \( m \mid (l-k) \), so \( l \equiv k \). Hence symmetric.

(iii) Assume \( k \equiv l \) and \( l \equiv r \). Then \( m \mid (k-l) \) and \( m \mid (l-r) \), so \( k-l = qm \) and \( l-r = pm \) for some \( q, p \in \mathbb{Z} \).

Adding these equations gives
\[
\begin{align*}
  k-l + l-r &= qm + pm \\
  k-r &= (q+p)m \text{ so } m \mid (k-r) \text{, so } k \equiv r.
\end{align*}
\]

Hence transitive.

For any \( k \in \mathbb{Z} \) we can write \( k = qm + r \), \( q, r \in \mathbb{Z} \) and \( 0 \leq r < m \). Thus \( k-r = qm \) and \( m \mid (k-r) \) so \( k \equiv r \).

So any \( k \) is congruent to one of the numbers \( \{0, 1, 2, \ldots, m-1\} \). On the other hand no two of these \( r \) are congruent, since if \( r, r' \in \{0, 1, 2, \ldots, m-1\} \), \( r \neq r' \), \( r-r' \) is too small in absolute value to be divisible by \( m \). So \( \{0, 1, 2, \ldots, m-1\} \) is a complete set of equivalence classes, and there are \( m \) of them.
6. For the following modes and tonic notes, indicate the appropriate key signature on the given staff, taking note of the clef:

(a) Phrygian with tonic D

(c) Aeolian with tonic G\textsuperscript{7}

7. Identify each chord in this major mode (Ionian) passage. Above the staff label each chord by root note class with suffix (e.g., B\textsuperscript{9}). Below the staff, label each chord by root scale tone (e.g. bIII\textsuperscript{7}).

8. Extend the following melody with two measures having the same rhythm, employing the following transformations. Do not write in a key change.

(a) diatonic up two scale tones in the second measure

(b) chromatic up a major third (from the original) in the third measure
9. Give the (total) duration in beats of:

(a) a doubly-dotted quarter note in $\frac{3}{2}$ time.

$$\frac{3}{2} \cdot \left(1 + \frac{1}{2} + \frac{1}{4}\right) = \frac{1}{2} \cdot \frac{7}{4} = \frac{7}{8}$$

(b) a sixteenth note in $\frac{9}{8}$ time (compound time signature).

$$\frac{1}{3} \cdot \frac{1}{16} = \frac{1}{48}$$

(c) a quarter note 5-tuplet in $\frac{4}{4}$ time.

$$k = \frac{5}{2^2 - 5 \leq 2^3 \leq 5} \quad \text{so} \quad n = 0 \quad \frac{1}{2^0} \text{ - note} = \text{whole note}$$

- $\Theta$ has $\frac{5}{4}$ beats

10. For the song *Mary Had A Little Lamb*, give the form (e.g., AABC) by dividing it into segments consisting of two bars. Locate and identify a translation other than that which comes from the overall form.

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Mary had a little lamb, little lamb, little lamb,
Mary had a little lamb, his fleece was white as snow.
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A  B  A'  C

melodic transcription m. 2, 3
rhythmic translation m. 2, 3, 4
also m. 1, 7
retrogression 1st 5 notes