REMINDER: We agreed in class that Exam II, a take-home exam, will be handed out in class on Thursday, November 3 and will be due on Tuesday, November 8. Between now and then, we'll have this assignment due Oct. 27, and Assignment #8 due Nov. 3. The style of the take-home will be very similar to that of homework assignments, some problems from the text, some made up by me. Good preparation for the take-home entails keeping up with the material discussed in class and continuing to work hard on homework problems.

I. Easy or relatively easy problems.

There are some even easier problems on these pages I didn't bother to assign. As a good warmup for the take-home exam, you should read over unassigned problems and see whether you can quickly provide a mental solution.

1. #33, p. 64

2. #13, p. 92. As mentioned in class, this was the proof of the extremal value theorem known to Weierstrass and his co-authors.

3. Generalize #14(b), p. 92, to show that, when \((E,d)\) has the property that each closed \(d\) — ball is compact, there is still a well-defined positive distance \(d(p_0,F)\) from any point \(p_0 \in E\) to a non-empty \(d\) — closed subset \(F \subset E\) not containing \(p_0\) and there exist points \(p \in F\) for which \(d(p_0,F)\) is equal to \(d(p, p_0)\). The condition that closed \(d\) — balls (meaning the closures of open \(d\) — balls) are compact implies that \((E,d)\) is locally compact but, contrary to an earlier version of this problem, it's unclear whether or not local compactness alone is enough to guarantee that the above properties for distances from points to closed sets.

4. (i) Use the same sort of approach as in #15 to show that, when \(K_1\) and \(K_2\) are compact in some metric space \((E,d)\), one can define a minimal separation distance \(d(K_1,K_2) = d(K_2,K_1)\) between the two sets with
$d(K_1, K_2) > 0 \iff K_1 \cap K_2 = \emptyset, \; d(p, q) \geq d(K_1, K_2) \forall (p, q) \in K_1 \times K_2$ and equality holds for at least one choice for $(p, q)$. Contrary to an earlier statement of this problem, this definition of separation distances between compact sets doesn't satisfy the triangle inequality; however, Hausdorff came up with a different definition of separation distances which does satisfy the triangle inequality.

(ii) For $(E, d) = (\mathbb{R}^n, d_2)$, generalize $(i)$ to the notion of a distance between a compact set and a closed set. If you are so inclined, generalize further to distances between compact and closed subsets of a metric space having the property in #3 that closed $d$-balls are compact.

(iii) Show by means of examples in $\mathbb{R}^2$ how $(ii)$ breaks down when we try to define distances between two closed, disjoint, non-compact sets. Specifically, give an example where the infimum of the distances between pairs of points in such sets is $0$ and give another example where the infimum is $> 0$ but isn't achieved by any pair of points.

II. Not so easy stereographic projection exercises.

In class, we defined the spherical metric $d_{sp}$ on the one-point compactification $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ of $\mathbb{R}^n$, $n \geq 2$, as the pull-back $\phi$ of the Euclidean metric on $S^n (= \text{unit } d_2\text{-sphere about 0 in } \mathbb{R}^{n+1})$ under the inverse of the stereographic projection map $\psi$. For $n = 2$, explicit calculations are greatly expedited by using complex numbers. Thus, $z = x + iy$ is just an alternative notation for $(x, y) \in \mathbb{R}^2$. For $p = (w, t)$ $\in S^2 \backslash \{(0, 1)\}$ (i.e., $w \in \mathbb{C}$ and $t \in [-1, 1]$) with $\|p\|_2^2 = 1 = |w|^2 + t^2$, $z = \psi(p) = \frac{z}{1-t}$ and $(z, 0) = p_0 + \frac{1}{1-t}(p - p_0)$ is the intersection point of $\mathbb{C} \times \{0\} \subset \mathbb{R}^3$ with the line in $\mathbb{R}^3$ through $p$ and $p_0$.

Easy to check is that $|z|^2 = \frac{1+t}{1-t}, \; t = \frac{|z|^2 - 1}{|z|^2 + 1}, \; 1 - t = \frac{2}{|z|^2 + 1}$, leading to $p = \psi^{-1}(z) = \phi(z) = (\frac{2z}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1})$. Also $\psi(p_0) = \infty$ and $\phi(\infty) = p_0$.

Then, for $z$ and $z'$ in $\overline{\mathbb{C}}$, $$(d_{sp}(z, z'))^2 = \|\phi(z) - \phi(z')\|_2^2 = 2 - 2 < \phi(z), \phi(z')> \text{ where } < \cdot, \cdot > \text{ is the standard inner product ("dot" product) on } \mathbb{C} \times \mathbb{R} = \mathbb{R}^3.$$ By a little bit of algebra, this works out to $d_{sp}(z, \infty) = \frac{2}{\sqrt{1+|z|^2}}$ for $z \in \mathbb{C}$ and $d_{sp}(z, z') = \frac{2|z - z'|}{\sqrt{(1+|z|^2)(1+|z'|^2)}}$ for $z, z'$ in $\mathbb{C}$. 
1. We say \( z_1 \) and \( z_2 \) are antipodal points in \( \overline{C} \) if \( d_{sph}(z_1, z_2) = 2 \); equivalently, this occurs \( \iff p_1 = \phi(z_1) \) and \( p_2 = \phi(z_2) \) are antipodal in \( S^2 \) in the usual sense that the line segment joining them is a common diameter of the family of great circles on \( S^2 \) passing through them \( \iff p_2 = -p_1 \). In particular, \( 0 \) and \( \infty \) are antipodal in \( \overline{C} \) since they correspond via \( \phi \) to the antipodal points \( p_0 \) and \( -p_0 \) in \( S^2 \). Give an explicit formula for the point \( z_2 \) which is antipodal to \( z_1 \) when \( z_1 \neq 0, \infty \).

2. Check that the image under \( \psi \) of the great circles passing through \( p_0 \) and \( -p_0 \) is the family of straight lines through the origin in \( \mathbb{C} \). Try to describe as explicitly as you can the family of curves through two antipodal points in \( \overline{C} \) which are the images under \( \psi \) of the family of great circles through the corresponding antipodal points in \( S^2 \). For this, you may find it helpful to write down the equations for these curves using \( d_{sph} \). Also, you should read over the hints to Problem 3 below before tackling this problem; using the notations in Problem 3, for which value for \( r \) are we describing a great circle in \( S^2 \)?

3. By a circle in \( \overline{C} \), we mean either an ordinary circle in \( \mathbb{C} = \mathbb{R}^2 \) or the 1-point compactification \( \mathbb{L} = \mathbb{L} \cup \{ \infty \} \) of an ordinary straight line in \( \mathbb{C} \). Show that the image under the stereographic projection map \( \psi \) of the family of circles in \( S^2 \) is the family of circles in \( \overline{C} \). For this, it's easiest to observe that the equation of a circle \( C' \) in \( S^2 \) can be described by \( \| p - q \|_2 = r \) where \( q \) is some fixed point in \( S^2 \), \( r \) is fixed in \((0, 2)\), and \( p \) is a variable point in \( S^2 \); then use the formula for \( d_{sph} \) to obtain an equation for the points \( z \in \psi(C') \) and show that this is the equation of a circle in \( \mathbb{C} \) when \( p_0 \notin C' \) and otherwise is the equation of the 1-point compactification of a line in \( \mathbb{C} \).

Caution: Because of the distortion involved in stereographic projection, for circles in \( S^2 \) not passing through \( p_0 \), the "center" \( q \) of the circle in \( S^2 \) doesn't always get mapped to the center of the image circle in \( \mathbb{C} \).

4. Show that the function from \( S^2 \) to \( S^2 \) which corresponds via stereographic projection to the inversion function \( f(z) = 1/z \) for \( z \notin \{0, \infty\} \), \( f(\infty) = 0 \), \( f(0) = \infty \) on \( \overline{C} \) is a rotation by the angle \( \pi \).
about a certain straight line through the origin in \( \mathbb{R}^3 \). Be sure to explicitly identify the straight line (=axis of rotation).

5. Extra credit for those who especially like Euclidean geometry, know a little bit about matrix groups, and are willing to make some lengthy calculations to express geometry by algebra.

A transformation \( T: \mathbb{C} \to \mathbb{C} \) is said to be a Möbius (or linear fractional) transformation if there is an invertible \( 2 \times 2 \) complex matrix
\[
A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}
\]
for which \( T(z) = T_A(z) = \frac{\alpha z + \beta}{\gamma z + \delta} \); when \( \gamma \neq 0 \), \( T_A(\infty) \) is understood to be \( \alpha/\gamma \) and \( T_A(-\delta/\gamma) \) is understood to be \( \infty \). Recall that \( A \) is invertible \( \iff \alpha \delta - \beta \gamma = \det(A) \neq 0 \). It's easy to see that \( T_A \circ T_B = T_{AB} \) where \( AB \) is the matrix product of the invertible matrices \( A \) and \( B \). Also \( T_{A'} = T_A \iff A' = cA \) for some non-zero complex number \( c \). This means that the collection of all Möbius transformations is a group under the composition operation and, modulo the changing of the sign of all entries, this group can be identified with the group of \( 2 \times 2 \) complex matrices having determinant equal to 1.

Use elementary group theory to show that the group of rotations on \( S^2 \) is in 1-1 correspondence via stereographic projection with the group of Möbius transformations on \( \mathbb{C} \) of the form \( T_A \) where \( A \) is a \( 2 \times 2 \) special unitary matrix, \( i.e. A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \) where \(|\alpha|^2 + |\beta|^2 = 1\). To do this, first show that, when \( \alpha = e^{i\theta} = \cos \theta + i \sin \theta \) and \( \beta = 0 \), \( T \) corresponds to a rotation by \( 2\theta \) about the axis in \( S^2 \) through \( p_0 \) and \( -p_0 \). Next look at the case \( \alpha = \cos \phi, \beta = \sin \phi \) for some \( \phi \in [0, \pi] \). Finally, do the general case by factoring \( A \) into the matrix product of a matrix of the first type, a matrix of the second type, and another matrix of the first type and using the fact that the composition of two rotations of \( S^2 \) is another rotation of \( S \).