Homework set 2 - Solutions
Math 3200 – Renato Feres

The first set of problems below is based on problem 2.39, page 70 in the textbook.

1. Using the sample function, produce a histogram for 100000 sample values of the demand random variable $X$. (The probabilities for $X = 1, 2, \ldots, 10$ are given in the statement of problem 2.39.) Use 10 bins, one for each value of $X$. Does your histogram show, approximately, the given probabilities for $X = x, x = 1, \ldots, 10$?

First define the probability vector

```r
demand.prob = c(0.01, 0.04, 0.05, 0.10, 0.15, 0.20, 0.20, 0.10, 0.10, 0.05)
```

Now generate the data:

```r
a = sample(1:10, 100000, replace=TRUE, prob=demand.prob)
```

The following commands create the desired histogram:

```r
hist(a, breaks=0.5+0:10, prob=TRUE, xlab="customer demand", 
+ main="Histogram of 100000 values of the demand random variable")
```

The values of the relative frequencies correspond fairly well with the exact probabilities.
2. Write an R function that gives the boy’s profit in terms of the number n of newspapers he orders and the customer demand x. (See part (b) of the problem 2.39.) Then use this to obtain the expected net profit for each value of n (say, for n from 1 to 10). For what n is the net profit a maximum?

The profit function can be written as follows:

```r
Profit=function(n,x){
  u=(1.5*x-n)*(x<=n)+(1.25*n-0.75*x)*(x>n)
  u
}
```

The probabilities for demand 1, 2, ..., 10 are

```r
demand.prob=c(0.01,0.04,0.05,0.10,0.15,0.20,0.20,0.10,0.10,0.05)
```

The values of the expected net profit for number ordered from 1 to 10 are obtained by

```r
n=10
e=0*(1:10) #This initializes e as a vector of 0s of length 10.
  #The expression e=array(0,c(1,10)) does the same.
  for(i in 1:n){
    e[i]=sum(Profit(i,1:10)*demand.prob)
  }
```

We obtain from this the following e:

```
-3.3925  -2.165  -1.0275  -0.0025   0.7975   1.26   1.2725  0.835  0.1725  -0.715
```

Thus the maximum happens for \( n = 7 \).

3. Plot the graph of the net profit as a function of n.

![Expected profit function graph](image)

This was obtained as follows:
plot(1:10,e,type='b',main="Expected profit function",
+ xlab="number of newspapers ordered",ylab="expected profit")
grid()

4. A probabilistic model of riffle shuffle commonly used in mathematics, known as the Gilbert-Shannon-Reeds model, consists of two steps: First we do a random cut (see the preliminaries) with the number \( k \) of cards in the top pile being a binomial random variable with parameters \( n = 52 \) and \( p = 0.5 \). Then we do a random interleaving operation as described on the homework set page.

Write in R a function \texttt{riffle.shuffle} which takes a vector \( a \) consisting of numbers from 1 to \( N \) (in any order), where \( N \) (the number of cards in our mathematical deck) is arbitrary, and produces a riffle shuffle of the vector. To test your program, apply it to the deck of 5 cards \( c(1, 2, 3, 4, 5) \).

Here is the program for a riffle-shuffle:

\begin{verbatim}
riffle.shuffle=function(a){
  n = length(a)
  k = rbinom(1,n,0.5)
  top = a[1:k]
  bottom = a[-(1:k)]
  e = sample(1:n,k)
  e = sort(e)
  b = a
  b[e] = top
  b[-e] = bottom
  b
}
\end{verbatim}

To test: \texttt{riffle.shuffle(c(1,2,3,4,5))} produces \( 3 \ 4 \ 1 \ 2 \ 5 \).

5. Now perform the following experiment. Each trial of the experiment consists of taking a virgin deck of 52 cards (meaning that it is in the order (1, 2, ..., 52)) then applying it 10 riffle-shuffle operations in succession, and finally recording the value of the top card of the shuffled deck. Repeat this experiment 10000 times, and produce a histogram (with 52 bins) of the data collected. (Thus the data consist of a vector containing 10000 numbers from 1 to 52.)

One expects that 10 riffle shuffle operations will produce a well-shuffled deck. Does your histogram support this expectation?

\begin{verbatim}
m=10000
top.card=array(0,m)
for(i in 1:m){
  a=52:1
  for(j in 1:10){
    a=riffle.shuffle(a)
  }
  top.card[i]=a[1]
}
\end{verbatim}
The histogram suggests to me that all cards have roughly the same probability of coming up at the top after 10 shuffles. This indicates that the deck is well-shuffled.