Preliminaries on Stokes' theorem and Riemannian manifolds

• Integration of forms. I review the basics about integrating an \(n\)-form over an \(n\)-dimensional manifold. Let \(M\) be a smooth \(n\)-manifold. We need \(M\) to be orientable. There are several equivalent characterizations of orientability; for example, \(M\) is orientable if there exists a nowhere vanishing continuous \(n\)-form \(\omega\). Given such an \(n\)-form we can select from a smooth atlas of \(M\) a subatlas for which the Jacobian determinant of coordinate changes are always positive. This subatlas can be defined by selecting only coordinates \(\varphi^{-1} = (x_1, \ldots, x_n)\) so that \(\omega(\partial/\partial x_1, \ldots, \partial/\partial x_n) > 0\). With the slight abuse of notation of thinking of \(\partial/\partial x_j\) as vectors either tangent to \(M\) or on \(\mathbb{R}^n\), this is the same as saying that \(\varphi^*\omega = \rho \, dx_1 \wedge \cdots \wedge dx_n\) where \(\rho\) is a positive real-valued function.

Let \(\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}\) be an oriented atlas of smooth parametrizations of \(M\), where \(U_\alpha \subset \mathbb{R}^n\) and \(\varphi_\alpha : U_\alpha \subset \mathbb{R}^n \to V_\alpha = \varphi_\alpha(U_\alpha) \subset M\) is a diffeomorphism for each \(\alpha\). We assume that \(\{V_\alpha : \alpha \in A\}\) is locally finite. This means that for each \(p \in M\) there is an open neighborhood of \(p\) that intersects at most finitely many \(V_\alpha\). Let \(\{\eta_\alpha : \alpha \in A\}\) be a smooth partition of unity subordinate to the open cover \(\{V_\alpha : \alpha \in A\}\). By definition, \(\{\eta_\alpha : \alpha \in A\}\) satisfies the properties:

1. \(\eta_\alpha : M \to [0,1]\) has compact support contained in \(V_\alpha\) for each \(\alpha\).
2. \(\sum_{\alpha \in A} \eta_\alpha(p) = 1\) for all \(p\). (The sum is finite due to the locally finite condition on the open cover.)

We have shown in class that partition of unities exist.

Definition 1. We assume the notations and definitions just given. In particular, let \(\{\eta_\alpha : \alpha \in A\}\) be a partition of unity subordinate to a locally finite open cover \(\{V_\alpha : \alpha \in A\}\) by images of parametrization maps \(\varphi_\alpha(U_\alpha) = V_\alpha\). Let \(\mu\) be a continuous \(n\)-form on \(M\) with compact support and write \(f_\alpha dx_1 \wedge \cdots \wedge dx_n = \varphi_\alpha^* (\eta_\alpha \mu)\). Then the integral of \(\mu\) on \(M\) is defined by

\[
\int_M \mu = \sum_\alpha \int_{U_\alpha} f_\alpha(x_1, \ldots, x_n) \, dx_1 \cdots dx_n.
\]

It is a simple verification that \(\int_M \mu\) is well-defined in that it does not depend on the choice of locally finite cover and on the choice of partition of unity. On the overlap of two parametrizations. In proving this fact we need to invoke the change of variables in integration identity: if \((y_1, \ldots, y_n) = h(x_1, \ldots, x_n)\) where \(h : U \to W\) is a (change of variables) diffeomorphism between open subsets of \(\mathbb{R}^n\), then

\[
\int_W f(y) \, dy_1 \cdots dy_n = \int_U f(h(x)) \left|\det(dh_x)\right| \, dx_1 \cdots dx_n.
\]

• Line integrals. A differentiable curve is a manifold of dimension 1 on which we may integrate one-forms. Integrals of one-forms are called line integrals. Let \(\gamma : [a, b] \to M\) be a differentiable parametric curve and \(\theta\) a
holds with $\Phi$. To verify property (3) first note that $j_H$ is a continuous one-form. Then
\[ \int_s^t \theta = \int_a^b \gamma^* \theta = \int_a^b \theta((\gamma'(t)) \, dt. \]
When $\theta = df$ we say that $\theta$ is an exact one-form. In this case
\[ \int_a^b df = df(\gamma'(t)) \, dt = \int_a^b \frac{d}{dt} f(\gamma(t)) \, dt = f(\gamma(b)) - f(\gamma(a)). \]
This means that the integral of an exact one-form only depends on the value of $f$ at the endpoints of the curve, and not on the curve itself. In particular, if $\gamma$ is a loop, so that $\gamma(a) = \gamma(b)$, then $\int_s^t df = 0$.

- **The Poincaré lemma.** A smooth $k$-form $\theta$ is said to be closed if $d\theta = 0$. Clearly an exact form $\theta = d\eta$ is closed because $dd = 0$. We wish to understand when a closed form is exact.

The manifold $M$ is said to be contractible if a smooth map $H : M \times \mathbb{R} \to M$ exists such that $H(p, 1) = p$ and $H(p, 0) = p_0$ for a fixed $p_0$ and all $p \in M$. For example, $M = \mathbb{R}^n$ is contractible, with $H(p, t) = tp$. A subset $U$ in a differentiable manifold, regarded as a smooth manifold itself, is contractible if it is diffeomorphic to $\mathbb{R}^n$.

**Theorem 2 (Poincaré’s lemma).** If a smooth manifold $M$ is contractible and $\theta$ is a smooth closed $k$-form, $k \geq 1$, then there exists a smooth $(k-1)$-form $\eta$ such that $\theta = d\eta$.

**Proof.** Let $N = M \times \mathbb{R}$ and define the flow $\Phi_t(p, s) = (p, s + t)$ on $N$. Let $Z$ be the vector field on $N$ associated to this flow. Thus we can identify $Z$ with the field $\frac{\partial}{\partial t}$ on $N$. Also define the map $j_t : M \to N$ by $j_t(p) = (p, t)$. Let $I$ be the map from smooth $k$-forms on $N$ to smooth $(k-1)$-forms on $M$ defined by
\[ (I(\xi)_p = \int_0^1 (j_t^*(i_Z \xi))_t \, dt. \]
Let $\xi$ be a smooth $k$-form on $N$ and define $\zeta = i_Z \xi$ and $\zeta_1 = \xi - dt \wedge \xi$. Then
1. $i_Z \xi = 0$, because $i_Z \circ \partial_Z = 0$.
2. $i_Z \xi_1 = 0$, because $i_Z(dt \wedge \xi) = dt(Z)\xi = \xi = i_Z \xi$.
3. $j_t^* \zeta - j_0^* \xi = dI \xi + 1d\xi$.

To verify property (3) first note that $\Phi_s \circ j_t = j_{t+s}$. Then
\[ dI \xi + 1d\xi = d \int_0^1 j_t^*(i_Z \xi) \, dt + \int_0^1 j_t^*(i_Z \xi) \, dt \]
\[ = \int_0^1 j_t^*(d(i_Z \xi) + i_Z d\xi) \, dt \]
\[ = \int_0^1 j_t^* \mathcal{L} Z \xi \, dt \]
\[ = \int_0^1 \frac{d}{ds} \bigg|_{s=0} (\Phi_s \circ j_t)^* \xi \, dt \]
\[ = \int_0^1 \frac{d}{dt} j_t^* \xi \, dt = j_0^* \xi - j_0^* \xi. \]

Now let $H : M \times \mathbb{R} \to M$ be a contraction map and define $\xi = H^* \theta$. Note that $d\xi = dH^* \theta = H^* d\theta = 0$. Also note that $j_t^* \xi = (H \circ j_t)^* \theta = id^* \theta = \theta$, and similarly, $j_0^* \xi = (H \circ j_0)^* \theta = 0$. Then $\theta = j_t^* \xi - j_0^* \xi = dI \xi$, so the theorem holds with $\eta = I \xi$. □
• **Stokes' theorem.** We need to extend our definition of manifolds so as to introduce boundaries. Define the half-space

\[ \mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \geq 0\}. \]

The half-space is the model of manifold with boundary just as \( \mathbb{R}^n \) is the model of an ordinary manifold. Manifolds with boundary are defined in terms of local parametrizations whose domains are now subsets of the half-space. This introduces two types of points on \( M \): the interior points, which lie in the image of a parametrization whose domain is contained in \( \mathbb{R}^n_+ \) minus the boundary (so that \( x_1 > 0 \)); and boundary points, which are the image under parametrization maps of points on the boundary of \( \mathbb{R}^n_+ \). This dichotomy between boundary points and interior points of a manifold with boundary rests on a non-trivial theorem in topology known as invariance of domain theorem. It can be stated as follows: If \( U \) is an open subset of \( \mathbb{R}^n \) and \( f : U \to \mathbb{R}^n \) is an injective continuous map, then \( V = f(U) \) is open and \( f \) is a homeomorphism between \( U \) and \( V \).

A function \( f \) defined on a subset of \( \mathbb{R}^n_+ \) is said to be differentiable at a point \( x \) on the boundary of \( \mathbb{R}^n_+ \) if \( f \) can be extended over a domain that includes an open neighborhood of \( x \) in \( \mathbb{R}^n \) and the extension is differentiable at \( x \) in the ordinary sense. Similarly, we can talk about smooth functions on regions in the half-space. With these preliminaries in mind, a smooth manifold with boundary is defined just as we did for regular manifolds but with \( \mathbb{R}^n_+ \) in place of \( \mathbb{R}^n \).

It is not difficult to show that the boundary \( \partial M \) of \( M \) is itself a smooth manifold of dimension \( n - 1 \). If \( M \) is orientable with positive orientation defined by some non-vanishing continuous \( n \)-form \( \omega \), then the boundary \( \partial M \) of \( M \) is also orientable. We define the positive boundary orientation to be the one given by the continuous \((n-1)\)-form \( \sigma = i_v \omega \), where \( v \) is any non-vanishing continuous vector field on \( \partial M \) such that for each \( p \in \partial M \) \( v(p) \in T_p M \) (but not in \( T_p(\partial M) \)) pointing out of \( M \). The notion of pointing “out of \( M \)” means that in any coordinate chart the image of \( v(p) \) in \( \mathbb{R}^n \) points into the negative \( x_1 \) half-space. This means that if \( d x_1 \wedge \cdots \wedge d x_n \) defines positive orientation on \( \mathbb{R}^n_+ \) then \( -d x_2 \wedge \cdots \wedge d x_n \) defines the induced positive orientation on the boundary of \( \mathbb{R}^n_+ \).

A basis \( \{u_1, \ldots, u_n\} \) of \( T_p M \) is said to be positive (relative to the orientation defined by the given non-vanishing continuous \( n \)-form \( \omega \)) if \( \omega_p(u_1, \ldots, u_n) > 0 \). With respect to the induced boundary orientation, a basis \( \{u_2, \ldots, u_n\} \) of \( T_p(\partial M) \) at a boundary point \( p \) is positive if \( \{u_1, u_2, \ldots, u_n\} \) is a positive basis of \( T_p M \) where \( u_1 \) is any outward pointing tangent vector to \( M \) at \( p \).

**Theorem 3** (Stokes’ theorem). Let \( M \) be an \( n \)-dimensional smooth, oriented, manifold with boundary \( \partial M \). Let
\(\mu\) be a differentiable \((n - 1)\)-form on \(M\) with compact support. We also denote by \(\mu\) the restriction of \(\mu\) to the boundary. (More formally, this restriction is \(i^*\mu\) where \(i : \partial M \to M\) is the inclusion map.) Then
\[
\int_{\partial M} \mu = \int_M d\mu.
\]

**Proof.** Using notation introduced above, let \(\eta_a : \alpha \in A\) be a partition of unity of \(M\) subordinate to the locally finite open cover \(\{V_\alpha : \alpha \in A\}\). On each coordinate chart with coordinates \((x_1, \ldots, x_n)\) we may write
\[
\eta_a \mu = \sum_{j=1}^n f_j \, dx_1 \wedge \cdots \wedge dx_j \wedge \cdots \wedge dx_n
\]
for certain functions \(f_j\) with compact support in \(\mathbb{R}^n\). The hat on top of \(dx_j\) indicates that this factor is omitted. By additivity of the integral we may as well assume \(\mu = \eta_a \mu\) and ignore the summation over \(\alpha\).

There are two types of coordinate neighborhoods: those parametrized over open sets in \(\mathbb{R}^n\) that do not intersect the boundary and those parametrized by open sets that contain boundary points of \(\mathbb{R}^n\). The argument given below using the fundamental theorem of calculus shows that the integral of \(\eta_a \mu\) over \(V_\alpha\) of the first kind is zero. So let us consider coordinate neighborhoods of the second kind.

Identifying for simplicity of notation \(d(\eta_a \mu)\) with its pull-back to \(\mathbb{R}^n\) under the coordinate (or parametrization) map we write
\[
d(\eta_a \mu) = \sum_{j=1}^n \frac{\partial f_j}{\partial x_j} \, dx_1 \wedge dx_1 \wedge \cdots \wedge dx_j \wedge \cdots \wedge dx_n = \sum_{j=1}^n (-1)^{j-1} \frac{\partial f_j}{\partial x_j} \, dx_1 \wedge \cdots \wedge dx_n.
\]

Note that
\[
\int_{\mathbb{R}^n} \frac{\partial f_j}{\partial x_j} \, dx_1 \wedge \cdots \wedge dx_n = 0
\]
if \(j \neq 1\). This is because, \(f_j\) being compactly supported,
\[
\int_{-\infty}^{\infty} \frac{\partial f_j}{\partial x_j} \, dx_j = 0
\]
for any fixed set of values of \(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n\). For \(j = 1\) on the other hand we have
\[
\int_{\mathbb{R}^n} \frac{\partial f_1}{\partial x_1} \, dx_1 \wedge \cdots \wedge dx_n = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( \int_{0}^{\infty} \frac{\partial f_1}{\partial x_1} \, dx_1 \right) \, dx_2 \ldots dx_n
\]
\[
= -\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_1(0, x_2, \ldots, x_n) \, dx_2 \ldots dx_n.
\]

Keeping in mind the orientation conventions for \(M\) and its boundary we have that the last integral in the above pair of equalities is the integral \(\int_{\partial M} \eta_a \mu\). This complete the proof. \(\square\)

**Riemannian manifolds.** Let \(M\) be a smooth manifold of dimension \(n\). A **Riemannian metric** on \(M\) is a symmetric, positive definite, symmetric tensor field \(g\) of type \((0, 2)\). In other words, \(g_p\) is a positive inner product at each \(p \in M\). We often write \(\langle u, v \rangle_p := g_p(u, v)\) or simply \(\langle u, v \rangle\) for \(u, v \in T_p M\). The Riemannian metric is smooth if the function \(p \mapsto \langle X(p), Y(p) \rangle\) is smooth for all smooth vector fields \(X, Y\). We call \(M\) together with a choice of Riemannian metric a **Riemannian manifold**.

Given a Riemannian metric on \(M\) and a vector \(v \in T_p M\), the **norm** of \(v\) is defined by
\[
\|v\|_p = \langle v, v \rangle^{1/2}.
\]

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The length of a differentiable curve $\gamma : [a, b] \to M$ is defined by
\[
\text{Length}(\gamma) = \int_a^b \|\gamma'(t)\| \, dt.
\]

A basis $\{u_1, \ldots, u_n\}$ of $T_pM$ is said to be orthonormal if $\langle u_i, u_j \rangle = \delta_{ij}$ where $\delta_{ij}$ are the entries of the identity matrix. A top-degree differential form $\omega$ on $M$ is called a volume form if for all $p \in M$ and every orthonormal basis $\{u_1, \ldots, u_n\}$ of $T_pM$
\[
\omega_p(u_1, \ldots, u_n) \in \{-1, 1\}
\]
holds. A Riemannian manifold is said to be orientable if it admits a volume form. (More generally, a manifold is said to be orientable if it admits a nowhere vanishing top-degree form.) The form itself defines an orientation of the manifold. Thus an orientable manifold has two possible orientations, which are specified by the choice of either $\omega$ or $-\omega$ as the volume form. If the orientation of $M$ is specified by $\omega$, then a basis $\{u_1, \ldots, u_n\}$ of $T_pM$, not necessarily orthonormal, is said to be positive if $\omega(u_1, \ldots, u_n) > 0$.

Here are some examples of Riemannian manifolds:

1. **Euclidean space.** This is the coordinate space $\mathbb{R}^n$ with the standard differentiable structure and endowed with the standard dot product Riemannian metric. By identifying the tangent space at each point with $\mathbb{R}^n$ itself and writing tangent vectors as $u = \sum_{i=1}^n u_i \frac{\partial}{\partial x_i}$ and $v = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}$, then
\[
\langle u, v \rangle_p = u \cdot v = \sum_{i=1}^n u_i v_i.
\]
The volume form in standard Cartesian coordinates is $\omega = dx_1 \wedge \cdots \wedge dx_n$.

2. **Conformally Euclidean metric.** The following modification of the dot product on Euclidean space is the source of a great variety of interesting examples of Riemannian manifolds. Let $\eta(x)$ be a positive real valued function on $\mathbb{R}^n$ and set
\[
\langle u, v \rangle_p = \eta(p)^2 u \cdot v.
\]
The vector fields $u_i = \frac{1}{\eta} \frac{\partial}{\partial x_i}$, $i = 1, \ldots, n$, make at each point $p$ an orthonormal basis of the tangent space of $\mathbb{R}^n$ at $p$. The one-forms $\theta_i = \eta dx_i$ form at each $p$ the dual basis, so that
\[
\omega = \eta^n dx_1 \wedge \cdots \wedge dx_n
\]
is the volume form.

Conformally Euclidean metrics are natural objects of study in geometric optics. Let $c$ denote the speed of light in vacuum, $v$ the speed of light in a not necessarily homogenous material medium, and $\eta = c/v$ the refractive index of the medium regarded as a function of position. One typically assumes $\eta \geq 1$ in this context but not always. Let $\gamma : (a, b) \to$ be a differentiable curve in space parametrized by (Euclidean) arclength $s$, representing the path of a light ray between two points. Let $t$ be time parameter. Then $v = \frac{ds}{dt}$, $d t = \frac{1}{c} \eta d s$, and
\[
T = \frac{1}{c} \int_a^b \eta(\gamma(s)) \, ds
\]
is the time it takes light to traverse this path. The quantity $S = cT$ is called the optical path length. Thus the length of a differentiable path in the conformally Euclidean metric equals the optical length of the path when the conformal factor $\eta$ is identified with the refraction index function of the optical medium. Fermat's principle of least time states that the path taken between two points by a ray of light is the path
that can be traversed in the least time. Equivalently, light rays (locally) minimize conformally Euclidean length for the conformal factor \( \eta \) given by the refractive index.

3. **Submanifolds of Euclidean space.** Let \( M \) be a smooth submanifold of \( \mathbb{R}^{n+m} \) of dimension \( n \). This means that \( M \) can be expressed, locally (that is, on an open neighborhood in \( \mathbb{R}^{n+m} \) of each of its points) as level set of a submersion from \( \mathbb{R}^{n+m} \) to \( \mathbb{R}^m \). It can also be expressed locally by an immersion from \( \mathbb{R}^n \) into \( \mathbb{R}^{n+m} \).

Recall that a submersion \( F \) is a differentiable map whose differential \( dF_p \) is surjective for each \( p \), and \( F \) is an immersion if \( dF_p \) is injective for each \( p \). When \( M \) is an \( n \)-dimensional submanifold of \( \mathbb{R}^{n+m} \), the tangent space \( T_p M \) may naturally be regarded as a linear subspace of \( \mathbb{R}^{n+m} \) and it makes sense to give \( M \) the induced Riemannian metric:

\[
\langle u, v \rangle_p = u \cdot v.
\]

More generally, \( M \) may be a smooth submanifold of some higher dimensional Riemannian manifold \( N \). The induced Riemannian metric on \( M \) is then defined at each \( p \) as the restriction to \( T_p M \) of the Riemannian inner product of \( T_p N \).

Consider the case of a smooth hypersurface \( M \) in \( \mathbb{R}^{n+1} \). This means that \( M \) is an \( n \)-dimensional smooth submanifold in Euclidean space of dimension one greater. At each \( p \in M \) there are exactly two vectors, \( \pm v(p) \) of unit length perpendicular to \( T_p M \). Let us assume that it is possible to choose a sign consistently so that \( p \in M \rightarrow v(p) \) is a smooth vector field. (This is not possible for the Möbius band, for example.) Existence of such a vector field is equivalent to \( M \) being orientable. Let \( \omega = dx_1 \wedge \cdots \wedge dx_{n+1} \) be the volume form on \( \mathbb{R}^{n+1} \). I claim that a volume form on \( M \) is given by

\[
\sigma_p = i_{v(p)} \omega_p.
\]

Recall that \( i_{v(p)} \) is the interior product. Thus if \( u_1, \ldots, u_n \) are vectors in \( T_p M \), then

\[
\sigma_p(u_1, \ldots, u_n) = \omega_p(v(p), u_1, \ldots, u_n).
\]

Is is clear that \( \sigma \) is a volume form on \( M \). This is because whenever \( u_1, \ldots, u_n \) form an orthonormal basis of \( T_p M \), then \( u_1, \ldots, u_n, v(p) \) is an orthonormal basis of \( \mathbb{R}^{n+1} \).

4. **Invariant metrics on Lie groups.** Let \( G \) be a Le group. Let \( L_g : G \rightarrow G \) denote left-translation by \( g \). We say that a Riemannian metric \( \langle \cdot, \cdot \rangle \) on \( G \) is left-invariant if

\[
\langle (dL_g)_h u, (dL_g)_h v \rangle_{gh} = \langle u, v \rangle_h
\]

for all \( u, v \in T_h G \). It is clear that a left-invariant Riemannian metric on \( G \) is completely determined by a positive inner product on the tangent space \( T_e G \) at the identity element since

\[
\langle u, v \rangle_g = \langle (dL_{g^{-1}})_e u, (dL_{g^{-1}})_e v \rangle_e.
\]

In particular, left-invariant Riemannian metrics always exist. Similar conclusions apply to right-invariant metrics. On the other hand, the existence of Riemannian metrics that are both left and right-invariant imposes strong conditions on \( G \). Clearly, abelian groups admit such bi-invariant metrics. It is not hard to show that compact groups also do.

5. **Kinetic energy metric.** Riemannian geometry is at the heart of classical (Newtonian) mechanics. The main point is that the kinetic energy associated to a mass distribution generally determines a Riemannian metric on the configuration manifold of a mechanical system. This can be explained with a couple of examples.
I. Constrained system of point masses. First consider a system consisting of two point masses in dimension $n$ such that mass $m_1$ is restricted to move in the hyperplane $\mathbb{R}^{n-1} = \{(x_1, \ldots, x_{n-1}, 0) \in \mathbb{R}^n\}$ and mass $m_2$ is restricted to move on the line $\mathbb{R} = \{(0, \ldots, 0, x_n) \in \mathbb{R}^n\}$. Let us impose the additional constraint that the two masses must always be at a distance 1 from each other. The configuration manifold for the two particles is therefore the unit sphere $S^{n-1}$. A motion of the system is then represented by a curve on the sphere. Define a Riemannian metric on $S^{n-1}$ given by

$$\langle u, v \rangle = m_1 \sum_{i=1}^{n-1} u_i v_i + m_2 u_n v_n$$

and denote its norm by $\| \cdot \|$. Then the kinetic energy of the state $(x, v)$, where $x \in S^n$ indicates configuration and velocity $v \in T_x S^{n-1}$, is

$$K(x, v) = \frac{1}{2} \| v \|^2.$$ 

It will be seen later that free motion of the system corresponds to a geodesic curve in the kinetic energy metric.

Note that the map $F: \mathbb{R}^n \to \mathbb{R}^n$ such that

$$F(x_1, \ldots, x_n) = \left(\sqrt{\frac{m_1}{m_1 + m_2}} x_1, \ldots, \sqrt{\frac{m_1}{m_1 + m_2}} x_{n-1}, \sqrt{\frac{m_2}{m_1 + m_2}} x_n\right)$$

maps the sphere diffeomorphically to an ellipsoid and pushes forward the Kinetic energy Riemannian metric on the sphere to the Riemannian metric on the ellipsoid induced as a submanifold of Euclidean space. Therefore, these two Riemannian manifolds are isometric.

II. Rigid bodies. Let a bounded region $\mathcal{B}$ of $\mathbb{R}^n$ represent a rigid body in some reference position in space, having density of mass function $\rho : \mathcal{B} \to [0, \infty)$. Being rigid means that any configuration of the body is given by an element of the Euclidean group $SE(n)$. Thus a configuration of the rigid body is a map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\varphi(x) = Ax + a$ where $a \in \mathbb{R}^n$ and $A \in SO(n)$. The kinetic energy of the body defines a Riemannian metric on $G = SE(n)$ as follows. Let $u, v \in T_g G$ and write $u = \frac{d}{dt}\big|_{t=0} ge^{t\zeta} x$, $v = \frac{d}{dt}\big|_{t=0} ge^{t\eta} x$ where $\zeta, \eta$ belong to the Lie algebra $se(n) = T_e G$. Note that for each material point $x \in \mathcal{B}$ the velocity of that point in (position-velocity) state $(g, x)$ is $\frac{d}{dt}\big|_{t=0} ge^{t\xi} x$. Now define the Riemannian metric on $G$ by

$$\langle u, v \rangle_G = \int_{\mathcal{B}} \left( \frac{d}{dt}\big|_{t=0} ge^{t\zeta} x \cdot \frac{d}{dt}\big|_{t=0} ge^{t\eta} x \right) \rho(x) \, dVol(x).$$

In particular,

$$K(g, v) := \frac{1}{2} \| v \|^2$$

is the kinetic energy of the rigid body in state $(g, v)$.

- Remarks on Maxwell’s equations. (This needs a lot of improvement.) Let $M$ be a 3-dimensional Riemannian manifold and $\omega$ its volume form and $\ast$ the Hodge star operator for the given Riemannian metric. The following introduces terminology and notation from electromagnetic theory. The point is simply to show how familiar equations look when written using differential forms and exterior calculus. In the below expressions $V$ indicates a three-dimensional bounded region with smooth boundary and $S$ smooth embedded surface in $M$ with smooth boundary curve.

1. $E$: Electric field intensity (1-form);
2. $B$: Magnetic field flux density (2-form);
3. $H$: Magnetic field intensity (1-form);
4. $D$: Electric field flux density (2-form);
5. $F = B + E \land dt$;
6. $G = D - H \land dt$;
7. $\rho$: Free charge density (0-form);
8. $j$: Free current density (2-form);
9. $J = \rho \omega - j \land dt$

\[
\int_{\partial S} E = -\frac{d}{dt} \int_{S} B \Rightarrow dE + \frac{\partial B}{\partial t} = 0 \quad \text{Faraday's law;}
\]
\[
\int_{\partial V} B = 0 \Rightarrow dB = 0 \quad \text{Gauss's law for magnetic charge;}
\]
\[
\int_{\partial S} H = \frac{d}{dt} \int_{S} D + 4\pi \int_{S} j \Rightarrow dH = \frac{\partial D}{\partial t} + 4\pi j \quad \text{Ampère's law;}
\]
\[
\int_{\partial V} D = 4\pi \int_{V} * \rho \Rightarrow dD = 4\pi * \rho \quad \text{Gauss's law;}
\]
\[
\int_{\partial V} j = -\frac{d}{dt} \int_{V} * \rho \Rightarrow * \frac{\partial \rho}{\partial t} = -d j \quad \text{Charge/current conservation.}
\]

The four of Mawell's equations can be written as one pair of equations using the fields $F$ and $G$ and source $J$ defined above. Here $d$ is the exterior derivative on the four-dimensional manifold $M \times \mathbb{R}$ where the $\mathbb{R}$ factor represents time. Thus, for example, if $A(x, t)$ is a $k$-form on $M \times \mathbb{R}$, and writing $d_{\omega}$ the exterior derivative operator on $M$, we have $dA = d_{\omega}A + \frac{\partial A}{\partial t} \land dt$. Then

\[
dF = 0
\]
\[
dG = 4\pi J.
\]

The fields $F$ and $G$ are related by $G = *F$ where $*$ is a relativistic Hodge-* operator. The equations look much more natural when expressed on a more general Lorentz manifold.

---

**Problems**

**Read all the problems given below and solve 4 of them.**

1. **Volumes of balls and spheres.** Let $S^n = \{ x \in \mathbb{R}^{n+1} : \| x \| = 1 \}$ be the $n$-dimensional sphere of radius 1 and $B^{n+1}(r)$ the $(n+1)$-dimensional ball of radius $r$, both centered at the origin of $\mathbb{R}^{n+1}$. The ball of radius 1 will be written $B^{n+1}$. Let $v(x) = x$ be the unit normal vector to the sphere at any $x \in S^n$ and write $\omega = dx_1 \land \cdots \land dx_{n+1}$ for the volume form in $\mathbb{R}^{n+1}$. The volume form on $S^n$ is then $\sigma = i_v \omega$ as explained in the introduction to this homework set. (I write $\sigma = \sigma^n$ when we need to be explicit about the dimension.) Define *polar coordinates* in $\mathbb{R}^{n+1} - \{0\}$ by the parametrization

$$
\varphi : (x, r) \in S^n \times (0, \infty) \mapsto r x \in \mathbb{R}^{n+1} - \{0\}
$$

(a) Show that $\varphi^* \omega = r^n dr \land \sigma$. 

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(b) Show that \( \text{Vol}(B^{n+1}) = \frac{1}{n+1} \text{Vol}(S^n) \). Denoting by \( S^n(r) \) the \( n \)-dimensional sphere of radius \( r \), conclude that

\[
\text{Vol}(S^n(r)) = \frac{d}{dr} \text{Vol}(B^{n+1}(r)).
\]

(c) Show that

\[
\text{Vol}(S^n) = \sqrt{n} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \text{Vol}(S^{n-1}) \quad \text{and} \quad \text{Vol}(S^n) = \frac{2^n n!}{n^{n+1}}.
\]

Suggestion: Consider the map \( \pi : S^n \to [-1, 1] \) defined by \( \pi(x_1, \ldots, x_{n+1}) = x_{n+1} \). Note that the level sets of \( \pi \) are spheres: \( \pi^{-1}(s) = S^n\left(\sqrt{1-s^2}\right) \). Define

\[
\psi : (x, s) \in S^{n-1} \times (-1, 1) \mapsto \left(\sqrt{1-s^2}x, s\right) \in S^n - \{N, S\}
\]

where \( N = (0, \ldots, 0, 1) \) and \( S = (0, \ldots, 0, -1) \) are the north and south poles of \( S^n \). Now prove that

\[
\psi^* \sigma^n = (1-s^2) \frac{n-2}{r} ds \wedge \sigma^{n-1}
\]

and conclude that

\[
\text{Vol}(S^n) = \text{Vol}(S^{n-1}) \int_{-1}^{1} (1-s^2) \frac{n-2}{r} ds.
\]

(d) Show that the ratio of the volume of a ball in dimension \( n = 100 \) by the volume of the \( n \)-dimensional cube (of side 2) in which the ball is inscribed is approximately \( 1.87 \times 10^{-70} \). This is the probability that a random point in the cube (with the uniform distribution) happens to be in the largest inscribed ball.

(Feel free to use Wolframalpha in any of your calculations.)

Solution. (a) Note that if \( u \) is any vector tangent to \( S^n \) at \( x \) and \( r \in (0, \infty) \), then

\[
d\varphi_{(x,u)} u = ru \quad \text{and} \quad d\varphi_{(x,u)} \frac{d}{dr} x = x.
\]

We know that the \((n+1)\)-form \( \varphi^* \omega \) must be a multiple of the top-degree form on \( S^n \times (0, \infty) \), so

\[
\varphi^* \omega = f(x, r) dr \wedge \sigma.
\]

For any given point \((x, r) \in S^n \times (0, \infty) \) let \( u_1, \ldots, u_n \) be an orthonormal basis of \( T_x S^n \). Then

\[
(\varphi^* \omega)(x, r) \left( \frac{d}{dr}, u_1, \ldots, u_n \right) = \omega_{rx} \left( d\varphi_{(x,r)} \frac{d}{dr}, d\varphi_{(x,r)} u_1, \ldots, d\varphi_{(x,r)} u_n \right) = r^n \omega_{rx}(x, u_1, \ldots, u_n) = r^n.
\]

On the other hand, \((dr \wedge \sigma) \left( \frac{d}{dr}, u_1, \ldots, u_n \right) = 1\). From this we conclude \( f(x, r) = r^n \) and \( \varphi^* \omega = r^n dr \wedge \sigma \) as claimed.

(b) The volume of the ball of radius \( r \) can be calculated in polar coordinates thus:

\[
\text{Vol}(B^{n+1}(r)) = \int_{B^{n+1}(r)} \omega = \int_{S^n \times (0, r)} \varphi^* \omega = \int_{S^n} \int_0^r s^n ds \wedge \sigma = \text{Vol}(S^n) \int_0^r s^n ds = \text{Vol}(S^n) \frac{r^{n+1}}{n+1}.
\]

When \( r = 1 \) we obtain \( \text{Vol}(B^{n+1}) = \frac{1}{n+1} \text{Vol}(S^n) \) as claimed. On the other hand, we know by simply scaling vol-
umes that \( \text{Vol}(S^n(r)) = r^n \text{Vol}(S^n) \). So

\[
\text{Vol}(S^n(r)) = r^n \text{Vol}(S^n) = \frac{d}{dr} \left( \text{Vol}(S^n) \frac{r^{n+1}}{n+1} \right) = \frac{d}{dr} \text{Vol}(B^{n+1}(r))
\]

(c) First note that \( d\psi_{(x,s)} u = \sqrt{1-s^2} u \) and

\[
d\psi_{(x,s)} \frac{d}{ds} = \frac{d}{ds} \left( \sqrt{1-s^2} x, s \right) = \left( -\frac{sx}{\sqrt{1-s^2}}, 1 \right).
\]

The latter vector is tangent to \( S^n \) at \( \psi(x,s) \) and perpendicular to the image of \( S^{n-1} \) under \( \psi(\cdot, s) \). (This is the sphere of radius \( \sqrt{1-s^2} \) at height \( s \) along the \( n \)th coordinate direction in \( \mathbb{R}^{n+1} \).) This vector has length \( 1/\sqrt{1-s^2} \). We denote it by \( \psi^* u_n = f(x,s)\sigma^{n-1} \wedge ds \) for some function \( f(x,s) \). Let \( u_1, \ldots, u_{n-1} \) be an orthonormal basis of the tangent space to \( S^{n-1} \) at \( x \) and \( d/ds \) the coordinate vector field on \( (-1,1) \). Then \( u_1, \ldots, u_n \) is an orthonormal basis of \( S^n \) at \( \psi(x,s) \) and

\[
f(x,s) = (\psi^* \sigma^n)_{(x,s)} \left( u_1, \ldots, u_{n-1}, \frac{d}{ds} \psi_{(x,s)} u_1, \ldots, \frac{d}{ds} \psi_{(x,s)} u_{n-1}, \frac{d}{ds} \psi_{(x,s)} u_n \right)
\]

\[
= (1-s^2)^{n-2} \sigma^n(u_1, \ldots, u_{n-1}, u_n) = (1-s^2)^{n-1}.
\]

This shows that \( \psi^* \sigma^n = (1-s^2)^{n-2} ds \wedge \sigma^{n-1} \) as claimed. Therefore,

\[
\text{Vol}(S^n) = \int_{-1}^{1} \int_{S^{n-1}} (1-s^2)^{n-2} \sigma^{n-1} \wedge ds = \text{Vol}(S^{n-1}) \int_{-1}^{1} (1-s^2)^{n-2}
\]

I plugged the integral into Wolfram alpha and obtained

\[
\text{Vol}(S^n) = \sqrt{\pi} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \text{Vol}(S^{n-1}).
\]

The expression

\[
\text{Vol}(S^n) = \frac{2\pi^{n+1}}{\Gamma\left(\frac{n+1}{2}\right)}
\]

is obtained by simple induction on the previous identity.

(d) This is a simple evaluation of the previous expression. Here I also used Wolfram alpha to get the number stated above.

\( \Diamond \)

2. **The divergence of a vector field.** Let \( X \) be a smooth vector field on an oriented Riemannian manifold \( M \) and \( \omega \) the volume form on \( M \). Define a function \( \text{div} X \) by

\[
\text{div}(X) \omega = \mathcal{L}_X \omega
\]

where \( \mathcal{L}_X \) is the Lie derivative. (Recall the formula \( \mathcal{L}_X = di_X + i_X d \).) The gradient vector field of a function \( f : M \to \mathbb{R} \) is defined by \( \nabla f = df \) where \( \nabla \) and its inverse map \( \nabla \) are the musical isomorphisms associated to the Riemannian inner product at each \( p \in M \).
(a) Show that if \( M = \mathbb{R}^n \) with the standard dot-product metric and \( X = \sum_j f_j \frac{\partial}{\partial x_j} \) then
\[
\text{div}(X) = \frac{\partial f_1}{\partial x_1} + \cdots + \frac{\partial f_n}{\partial x_n}.
\]

(b) Show that \( \text{div} X = *d * X^\flat \) where * is the Hodge-star operator.

(c) Show that the Laplacian \( \Delta f = \text{div} \text{grad} f = *d * d f \) and if \( M \) is Euclidean space then
\[
\Delta f = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}.
\]

(Note to physics students: The symbol \( \nabla \) will later be used to denote covariant differentiation, in which case \( \nabla^2 f \) will represent a \((0,2)\)-tensor. The Laplacian will then be the trace of the \((1,1)\)-tensor obtained from \( \nabla^2 f \) via the \( \sharp \) operator. The notation \( \Delta \) for the Laplacian is standard in mathematics.)

**Solution.** (a) Note that \( \mathcal{L}_X \omega = (di_X + i_X d)\omega = di_X \omega \). Now
\[
i_X \omega = i_X dx_1 \wedge \cdots \wedge dx_n = \sum_{i=1}^n (-1)^{i-1} dx_1 \wedge \cdots \wedge i_X dx_i \wedge \cdots \wedge dx_n = \sum_{i=1}^n (-1)^{i-1} f_i dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n
\]
and
\[
di_X \omega = \sum_{i=1}^n (-1)^{i-1} df_i \wedge dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n
\]
\[
= \sum_{i=1}^n df_i \wedge dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n
\]
\[
= \left( \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} \right) dx_1 \wedge \cdots \wedge dx_n = \left( \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} \right) \omega.
\]
Therefore \( \text{div}(X) = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} \).

(b) First observe that
\[
* dx_j = (-1)^{j-1} dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_n
\]
so
\[
* d * X^\flat = \sum_{j=1}^n (-1)^{j-1} df_j \wedge dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_n = \left( \sum_{j=1}^n \frac{\partial f_j}{\partial x_j} \right) * (dx_1 \wedge \cdots \wedge dx_n) = \sum_{j=1}^n \frac{\partial f_j}{\partial x_j}.
\]
Therefore \( * d * X^\flat = \text{div}(X) \).

(c) If \( X = \text{grad} f \) then \( X^\flat = df \) and \( \text{div} \text{grad} f = *d * df \). The sum of the second derivatives expression is now immediate.

\[\diamondsuit\]

3. **Relations between div, grad and curl.** Let \( M \) be a 3-dimensional Riemannian manifold. If \( f \) is a smooth function on \( M \) and \( X \) is a smooth vector field on \( M \) we define
\[
\text{grad} f = (df)^\flat, \quad \text{div} X = *d * X^\flat, \quad \text{curl} X = \left( *d X^\flat \right)^\sharp.
\]

Using Poincaré’s lemma show that
(a) If \( \text{curl} \, X = 0 \) then on any contractible open subset of \( M \) we have \( X = \text{grad} \, h \) for some smooth function \( h \).

(b) If \( \text{div} \, X = 0 \) then on any contractible open subset of \( M \) we have \( X = \text{curl} \, Y \) for some smooth vector field \( Y \).

**Solution.** First note that \( * \, v = v \) for all forms in dimension 3.

(a) If \( \text{curl} \, X = 0 \) then \( d \, X^\flat = 0 \) because \( \sharp \) and \( * \) are invertible operations. By Poincaré’s lemma this implies that on any contractible subset of \( M \) there exists a function \( f \) such that \( X^\flat = d \, f \) so \( X = \text{grad} \, f \).

(b) If \( \text{div} \, X = 0 \) then \( d \, * \, X^\flat = 0 \) because \( * \) is invertible. By Poincaré’s lemma \( * \, X^\flat = d \, \nu \) for some 1-form \( \nu \). Let \( Y = \nu \# \). Then \( X = ( \ast d Y )^\sharp \).

\( \diamond \)

4. **Green’s theorem.** Let \( f(x, y) \) and \( g(x, y) \) be smooth functions on a domain \( \mathcal{R} \) in \( \mathbb{R}^2 \) bounded by a simple closed curve \( \partial \mathcal{R} \). Show that Green’s formula

\[ \int_{\mathcal{R}} (\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}) \, dx \, dy = \int_{\partial \mathcal{R}} f \, dx + g \, dy \]

can be derived from the general Stokes’ theorem.

**Solution.** The key point is simply that

\[ d(f \, dx + g \, dy) = df \wedge dx + dg \wedge dy = \frac{\partial f}{\partial y} \, dy \wedge dx + \frac{\partial g}{\partial x} \, dx \wedge dy = \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \, dx \wedge dy. \]

Therefore, Green’s theorem follows directly from Stokes’ theorem.

\( \diamond \)

5. **The divergence theorem.** Let \( M \) be a Riemannian manifold with boundary and volume form \( \omega \). Denote by \( \nu \) the outward pointing unit normal vector field on the boundary of \( M \) and let \( \sigma := \iota_\nu \omega \) be the volume form on \( \partial M \). Let \( X \) be a compactly supported smooth vector field on \( M \). Show that the equality

\[ \int_M \text{div}(X) \, \omega = \int_{\partial M} \langle X, \nu \rangle \, \sigma \]

follows from the general Stokes’s theorem. In words, the flux of \( X \) across the boundary of \( M \) equals the volumetric integral of the divergence of \( X \).

**Solution.** First note that the restriction of \( \iota_X \omega \) to the boundary of \( M \) is a top-degree form on \( \partial M \) and so \( \iota_X \omega = h \sigma \) for some function \( h \). To obtain \( h \) we evaluate the form on an orthonormal basis \( u_1, \ldots, u_{n-1} \) of a tangent space to \( \partial M \). This gives

\[ h = h \sigma(u_1, \ldots, u_{n-1}) = \omega(X, u_1, \ldots, u_n) = \langle X, \nu \rangle \omega(\nu, u_1, \ldots, u_n) = \langle X, \nu \rangle. \]

Thus we have

\[ \int_M \text{div}(X) \omega = \int_M d \iota_X \omega = \int_{\partial M} \iota_X \omega = \int_{\partial M} \langle X, \nu \rangle \sigma. \]

\( \diamond \)

6. **Kinetic energy metric for the rigid body.** Consider the Riemannian metric on the group \( SE(n) \) defined by the identity 1.
(a) Show that this metric is left-invariant.

(b) Find a coordinate expression for the metric on $SE(2)$ associated to a disc of radius $R$ with uniform mass distribution. Use coordinates $(x, y, \theta)$ where $(x, y)$ represents the Cartesian coordinates of the center of the disc and $\theta$ the angle of rotation.

**Solution.** (a) Here I use notation and facts from the homework set number 1. I denote and not on the point $g$ where $m$ is the mass of the body. Quite explicitly, the expression only depends on the Lie algebra elements $\xi, \eta$ and not on the point $g$. In fact, we have just shown that

\[
\langle (dL_g)_e \xi, (dL_g)_e \eta \rangle = \langle \xi, \eta \rangle
\]

where $L_g$ is left-translation by $g$ on the Euclidean group. But this means that the metric is left-invariant.

(b) I will use $(x_1, x_2)$ for the $(x, y)$-coordinates on the body $\mathcal{B}$, leaving $(x, y, \theta)$ for the coordinates on the configuration manifold $SE(2)$, which is topologically $\mathbb{R}^2 \times S^1$. Thus, for the configuration $q = (x, y, \theta)$ and tangent vector $u = (\dot{x}, \dot{y}, \dot{\theta})$ at $q$, the material point $(x_1, x_2) \in \mathcal{B}$ will be at position

\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
+ \begin{pmatrix}
x \\
y
\end{pmatrix}
\]

with the velocity

\[
\dot{\theta} \begin{pmatrix}
-\sin \theta & -\cos \theta \\
\cos \theta & -\sin \theta
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
+ \begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix}.
\]

Note that

\[
A = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}, \quad Z = \dot{\theta} \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}, \quad z = \begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix}
\]

so that

\[
A(Zx + z) = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
\dot{\theta} \begin{pmatrix}
-x_2 \\
x_1
\end{pmatrix}
+ \begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix}
\end{pmatrix}
\]

If $u_i = (\dot{x}_i, \dot{y}_i, \dot{\theta}_i)$ for $i = 1, 2$ are two tangent vectors at $(x, y, \theta)$, then from expression 2 we have

\[
\langle u_1, u_2 \rangle_q = \frac{m}{\pi R^2} \dot{\theta}_1 \dot{\theta}_2 \int_{\mathcal{B}} \left( x_1^2 + x_2^2 \right) dA(x_1, x_2) + m(\dot{x}_1 x_2 + \dot{y}_1 y_2).
\]

The area integral, easily obtained by change to polar coordinates, has value $\pi R^2 / 2$. Therefore,

\[
\langle u_1, u_2 \rangle_q = \frac{1}{2} m R^2 \dot{\theta}_1 \dot{\theta}_2 + m(\dot{x}_1 x_2 + \dot{y}_1 y_2).
\]
Note that this metric corresponds to the kinetic energy

\[ K(\dot{x}, \dot{y}, \dot{\theta}) = \frac{m}{2} \left( \frac{1}{2} R^2 \dot{\theta}^2 + \dot{x}^2 + \dot{y}^2 \right). \]

Also note that by defining the new coordinate \( z = r \theta / \sqrt{2} \) (unrelated to the previously defined \( z \)) the metric becomes a constant multiple of the Euclidean metric in \( \mathbb{R}^3 \).

\[ \diamond \]

7. **Maxwell’s equations.** Derive the pair of Maxwell’s equations \( dF = 0 \) and \( dG = 4\pi J \) from the four equations given in terms of \( H, B, E, D \).

**Solution.** For \( F \) we use Maxwell’s equations \( d_\omega B = 0 \) and \( d_\omega E = - \frac{\partial B}{\partial t} \) and find

\[
dF = d_\omega B + \frac{\partial B}{\partial t} \wedge dt + d_\omega E \wedge dt = \frac{\partial B}{\partial t} \wedge dt - \frac{\partial B}{\partial t} \wedge dt = 0.
\]

For \( G \) we use Maxwell’s equations \( d_\omega D = 4\pi \rho \) and \( d_\omega H = \frac{\partial D}{\partial t} + 4\pi j \) and find

\[
dG = 4\pi \rho + \frac{\partial D}{\partial t} \wedge dt - \left( \frac{\partial D}{\partial t} + 4\pi j \right) \wedge dt = 4\pi (\rho \wedge dt) = 4\pi J.
\]

\[ \diamond \]