Covariant differentiation

Homework set 9 - due 04/13/2015

Math 407 – Renato Feres

Covariant differentiation

- Connections in \( \mathbb{R}^n \). We have by now defined a number of different notions of differentiation: the differential of maps between manifolds, the exterior derivative of differential forms, and the Lie derivative. To begin to see the need for a new notion of differentiation (which will turn out to be the most fundamental and will apply to general tensor fields) consider first the problem of defining the directional derivative of a vector field \( X \) in \( V = \mathbb{R}^n \). Let \( v \in T_p V \). Observe that the obvious definition

\[
D_v X = \lim_{t \to 0} \frac{X(p + tv) - X(p)}{t}
\]

involves taking the difference between two vectors that live in different vector spaces: \( T_{p+tv} V \) and \( T_p V \). The definition makes sense because \( V \) is a vector space and these two spaces are naturally identified with \( V \) itself by translation. When we move from a vector space \( V \) to a general manifold some way of connecting tangent spaces at two distinct points is needed. This is already an issue in trying to define the acceleration of a curve, which is the time-derivative of the velocity vector field along the curve. The velocity vector field of a smooth path makes sense on a general manifold, but we do not yet have a way of defining acceleration based on the manifold smooth structure alone; some additional structure is needed. (Note that for the Lie derivative \( \mathcal{L}_X \) the differential of the flow of \( X \) is used to carry the vector or tensor from one tangent space to another.)

Let us begin by isolating the main properties of the Euclidean \( D \) and then use what we learn to define a general notion of directional derivative. (Here and in many other texts the words covariant differentiation and connection are used more or less interchangeably.)

For simplicity of notation I write \( \partial_i = \partial/\partial x_i \). Let \( X = h_1 \partial_i + \cdots + h_n \partial_n \) be a smooth vector field on \( V \) and \( v \in V_p \) a tangent vector at \( p \). We take the following characterization of \( D \) as our starting point.

\[
D_v X = \sum_{i=1}^n (v_p h_i) \partial_i. \quad (1)
\]

Note, in particular, that \( D_v X = 0 \) for all \( p \) and all \( v \) if and only if the components \( h_i \) of \( X \) are constant. The main properties of \( D \) are stated below in Proposition 1. You’ll prove the proposition as an exercise.

**Proposition 1.** The following properties hold for the derivative of vector fields operation \( D \) defined in 1:

\begin{align*}
(c_1) \quad D_v (X + Y) &= D_v X + D_v Y \\
(c_2) \quad D_v (f X) &= (v f) X + f D_v X \\
(c_3) \quad D_{au+bv} X &= a D_u X + b D_v X
\end{align*}

The above three properties define a covariant derivative of vector fields. In addition:
(c₄) \( D_X Y - D_Y X = [X, Y] \)

(c₅) \( v_p(X, Y) = (D_v X, Y) + (X, D_v Y) \)

where \((\cdot, \cdot)\) denotes dot product. These two additional properties define the (Euclidean) Levi-Civita covariant derivative. The above five properties fully characterize \(D\). Finally,

(c₆) \( |D_X D_Y| Z = D_{[X,Y]} Z \). This last property says that \(D\) is flat. (More on this and the concept of curvature later.)

• Connections on general manifolds. Let \(M\) be a smooth manifold. We define a connection (on \(TM\)) by the first three properties of Proposition 1.

**Definition 2** (Covariant differentiation of vector fields). A differential operator \(\nabla\) that takes \(v \in T_pM\) and a smooth vector field \(X\) and associates a vector \(\nabla_v X \in T_pM\) is called a (linear) connection, or covariant derivative of vector fields, if for all \(u, v \in T_pM\), smooth function \(f : M \to \mathbb{R}\), vector fields \(X, Y\) and real numbers \(a, b\)

1. \(\nabla_{au+bv} X = a \nabla_u X + b \nabla_v X\)
2. \(\nabla_v (X + Y) = \nabla_v X + \nabla_v Y\)
3. \(\nabla_v (fX) = (vf)X + f \nabla_v X\).

The operator \(\nabla\) is smooth if \(\nabla_X Y\) is a smooth vector field whenever \(X, Y\) are smooth vector fields. It is said to be a torsion-free connection if

(c₄) \(\nabla_X Y - \nabla_Y X = [X, Y]\).

If \(M\) is a Riemannian manifold with Riemannian metric \((\cdot, \cdot)\), then \(\nabla\) is said to be a metric or Riemannian connection if the following product rule holds:

(c₅) \(v_p(X, Y) = (\nabla_v X, Y) + (X, \nabla_v Y)\)

The connection (not necessarily metric) is said to be flat if

(c₆) \(\nabla_X \nabla_Y Z = \nabla_{[X,Y]} Z\) for all smooth vector fields \(X, Y, Z\).

A number of observations will help to put some flesh in this formal definition. Let us begin by seeing how the choice of \(\nabla\) (satisfying only \(c₁, c₂, c₃\)) allows to define a notion of *acceleration*. We first need to extend the definition of \(\nabla_v X\) from vector fields on \(M\) to vector fields only defined along a differentiable path.

Let \(t \to X(t)\) be a vector field defined along a differentiable path \(\gamma : [a, b] \to M\). This means that \(X(t) \in T_{\gamma(t)} M\) for each \(t\). Note that \(X\) does not have to be the restriction of a vector field defined on all of \(M\) (or even on some neighborhood of the path) to the image of \(\gamma\). One example is the velocity vector field of \(\gamma\). Note that if \(\gamma\) intersects itself, say \(\gamma(s) = \gamma(t)\), and if \(\gamma'(s) \neq \gamma'(t)\), then \(X(t) = \gamma'(t)\) cannot extend to a vector field on a neighborhood of the curve.

**Proposition 3** (Covariant derivative along curves). Let \(M\) be a smooth manifold with connection \(\nabla\). Then there exists a unique correspondence that associates to each vector field \(X(t)\) along a differentiable path \(c(t), t \in [a, b]\), a vector field \(\frac{\delta X}{\delta t}\) along \(\gamma\), called the covariant derivative of \(X\) along \(\gamma\), such that for vector fields \(X, Y\) along \(\gamma\) and \(f\) a differentiable function on \([a, b]\)

1. \(\frac{\delta}{\delta t}(X + Y) = \frac{\delta X}{\delta t} + \frac{\delta Y}{\delta t}\)
2. \(\frac{\delta}{\delta t}(fX) = f \frac{\delta}{\delta t}X + \frac{\delta f}{\delta t} X\)
3. If \(X(t) = X(\gamma(t))\) is the restriction to \(\gamma\) of a vector field \(X\) on \(M\) then \(\frac{\delta X}{\delta t} = \nabla_{\gamma'(t)} X\)
4. If the connection is Riemannian,

\[
\frac{d}{dt} \langle X, Y \rangle = \left\langle \frac{\nabla X}{dt}, Y \right\rangle + \left\langle X, \frac{\nabla Y}{dt} \right\rangle.
\]

**Proof.** The main point is to observe that \( \frac{\nabla X}{dt} \) satisfying the above three conditions must have the following local expression in terms of a local frame of vector fields. By a frame I mean a collection \( X_1, \ldots, X_n \) of smooth vector fields defined on some neighborhood of \( \gamma(t) \) that are linearly independent at each point of that neighborhood. Then we can write \( X(t) = f_1(t)X_1(\gamma(t)) + \cdots + f_n(t)X_n(\gamma(t)) \) and \( \gamma'(t) = v_1(t)X_1(\gamma(t)) + \cdots + v_n(t)X_n(\gamma(t)) \). It follows from the three properties that

\[
\frac{\nabla X}{dt} = \sum_i \frac{df_i}{dt} X_j + \sum_{i,j} f_j v_i \nabla X_i X_j.
\]

This shows existence. The claim about Riemannian connections follows immediately from the definition. \( \square \)

**Definition 4** (Acceleration and geodesics). Let \( M \) be equipped with a choice of connection \( \nabla \). The acceleration of \( \gamma(t) \) relative to \( \nabla \) is the vector field along \( \gamma \) given by \( \frac{\nabla \gamma'}{dt} \). The curve is said to be a geodesic relative to \( \nabla \) if \( \frac{\nabla \gamma'}{dt} = 0 \) for all \( t \).

In a mechanical setting, a choice of connection defines what we may want to understand by “natural motion,” namely, that kind of motion represented by a curve \( \gamma \) (in the configuration manifold of the system) that is not subject to any forces. Forces are what cause \( \gamma \) to deviate from natural motion. Thus it makes sense to regard a force field acting on the system as a vector field \( F(t) \) such that \( \frac{\nabla F}{dt} = F \). This is one form of Newton's equation. In this form it is understood that \( \nabla \) carries information about the mass distribution of the system. More on mechanical applications later. Note that the definition of geodesic curves as curves with zero acceleration makes no use of metric concepts such as length. The connection with (local) length minimization will be made later in the context of Riemannian manifolds.

**Definition 5** (Parallel transport). A vector field \( X(t) \) along a differentiable curve \( \gamma : [a, b] \to M \) is said to be parallel relative to a connection \( \nabla \) if \( \frac{\nabla X}{dt} = 0 \).

For example, if \( X(t) = \sum_i h_i(t) \partial / \partial x_i \) is a parallel vector field along a curve \( \gamma(t) \) in \( \mathbb{R}^n \) relative to the Euclidean connection \( D \), then \( X \) is a constant vector field; that is, the \( h_i \) are constant. This means that parallel translation in this case amounts simply to vector space translation of the vector from one point to another. In particular, the notion does not depend on the path joining those points. Also observe that a geodesic is a curve whose velocity field is parallel.

**Proposition 6** (The parallel translation map). Let \( M \) be a smooth manifold with connection \( \nabla \) and \( \gamma : [a, b] \to M \) a differentiable path joining \( p_1 = \gamma(a) \) and \( p_2 = \gamma(b) \). Then for each \( v \in T_{p_1} M \) there exists a unique parallel vector field \( X(t) \) along \( \gamma \) such that \( X(0) = v \). The vector \( X(b) \in T_{p_2} M \) is a linear function of the initial vector \( v \), giving...
rise to a linear map $P_\gamma : T_{p_1}M \to T_{p_2}M$ called the parallel translation map. These maps satisfy the following properties, in which $\gamma^{-1}$ denotes the path $\gamma$ traversed backward and $\gamma_2\gamma_1$ denotes concatenation of paths:

1. $P_{\gamma^{-1}} = P_\gamma^{-1}$
2. $P_{\gamma_2\gamma_1} = P_{\gamma_2} \circ P_{\gamma_1}$
3. $P_\gamma$ is an orthogonal map if $M$ is equipped with a Riemannian metric and the connection is metric (Riemannian) with respect to this metric.
4. The set of all $P_\gamma$ where $\gamma$ is a piecewise differentiable closed curve beginning and ending at $p \in M$ constitutes a subgroup of $GL(T_pM)$ called the holonomy group of $\nabla$ at $p$. If $\nabla$ is a metric connection, the holonomy group is contained in $O(T_pM)$.

The covariant derivative can be recovered from the parallel translation maps by

$$\nabla \frac{X}{dt} = \lim_{t \to 0} P_{\gamma(t)}^{-1} \frac{X(\gamma(t)) - X(p)}{t}$$

where $\gamma$ is the path $\gamma : [0, a] \to M$ restricted to $[0, t], t \leq a$.

**Proof.** Let $\gamma(t)$ be a differentiable path, $0 \leq t \leq 1$, and $X_1(t), \ldots, X_n(t)$ smooth vector fields along $\gamma(t)$ forming a basis of $T_{\gamma(t)}M$ for each $t$. Let us write $\frac{\nabla X_i}{dt} = \sum_j H_{ij} X_j$ where $H_{ij}(t)$ is a differentiable function. A vector field $X(t)$ along $\gamma$ may be written as a linear combination $X = \sum_j h_j X_j$. Then

$$\nabla \frac{X}{dt} = \sum_j h'_j X_j + \sum_{i,j} h_j H_{ij} X_i.$$  

This means that $X$ is parallel along $\gamma$ if and only if the coefficients $h_i$ are the solutions of the system of linear equations

$$h'_i = \sum_j H_{ij} h_j, \quad i = 1, \ldots, n.$$  

By the existence and uniqueness theorem for systems of (linear) ordinary differential equations we obtain a solution vector $h(t) \in \mathbb{R}^n$ such that $h(t)$ is the vector of coefficients of $v$ in the basis $\{X_1(0), \ldots, X_n(0)\}$. By linearity of the differential equation it is clear that if $g(t), h(t)$ are two solutions of the system corresponding to initial vectors $u$ and $v$, then $ag(t) + bh(t)$ is the solution with initial vector $au + bv$. Therefore, the correspondence

$$\sum_j h_j(0) X_j(0) \to \sum_j h_j(1) X_j(1)$$

is the linear map $P_\gamma$ from $T_{\gamma(0)}M$ to $T_{\gamma(1)}M$. The claimed properties all follow from these basic facts about systems of ordinary differential equations. In particular, let us suppose that the $X_j(t)$ are parallel fields. Then $\nabla X_j / dt = 0$, $X_j(t) = P_\gamma X_j(0)$ and

$$\lim_{t \to 0} \frac{P_{\gamma(t)}^{-1} \frac{X(\gamma(t)) - X(0)}{t}}{t} = \lim_{t \to 0} \sum_j \frac{h_j(t) - h_j(0)}{t} X_j(0) = \sum_j \frac{h_j'(0)}{t} X_j(0) = \nabla \frac{X}{dt} \big|_{t=0}.$$  

This shows how the covariant differentiation can be recovered from parallel transport. If the connection is Riemannian,

$$\frac{d}{dt} \langle P_\gamma v, P_\gamma w \rangle = \left\langle \frac{\nabla}{dt} P_\gamma v, P_\gamma w \right\rangle + \left\langle P_\gamma v, \frac{\nabla}{dt} P_\gamma w \right\rangle = 0.$$  

\[\square\]
Two tensor fields associated to $\nabla$ should be highlighted.

**Proposition 7.** Given a connection $\nabla$ on the smooth manifold $M$ there exist tensor fields $T$ of type $(1,2)$ and $R$ of type $(1,3)$ such that for any smooth vector fields $X, Y, Z$ on $M$

$$T(X_p, Y_p) = \left(\nabla_X Y - \nabla_Y X - [X, Y]\right)_p$$

and

$$R(X_p, Y_p) Z_p = \left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\right)_p.$$

In particular, these quantities a priori defined in terms of derivatives of the vector fields in fact only depend on the values of these vector fields at $p$.

**Proof.** To show $T$ is indeed a tensor field we need to check that $T(f X, Y) = f T(X, Y) = T(X, f Y)$ for all smooth functions $f$. This is a straightforward consequence of the definition of $T$. Similarly, one easily checks that $R(f X, Y) Z = f R(X, Y) Z = R(X, f Y) Z = R(X, Y)(f Z)$.

**Definition 8** (Torsion tensor $T$ and the Riemann curvature tensor $R$). The tensor field $T$ is called the torsion tensor of $\nabla$ and $R$ is called the Riemann curvature tensor of $\nabla$. When $T$ is identically zero we say that $\nabla$ is torsion-free and when $R$ is identically zero we say that $\nabla$ is flat.

As already noted, the standard connection $\nabla$ in $\mathbb{R}^n$ is both torsion-free and flat (as well as metric). In fact, $T$ and $R$ are precisely the obstructions (when not zero) for a manifold $M$ with connection $\nabla$ to be locally like $(\mathbb{R}^n, \nabla)$. More precisely, if $R$ and $T$ are both zero, then its is possible to find local coordinates around any given point $p \in M$ so that, in these coordinates, geodesics are straight lines paths with constant speed, and parallel transport is vector translation in $\mathbb{R}^n$. The following theorem is somewhat more involved than the other ones in these notes; I state it here with only a sketch of proof.

**Theorem 9.** Let $\nabla$ be a connection on a manifold $M$ having zero torsion and zero curvature tensor. Then around each point we can find a local parametrization $\varphi : U \subset \mathbb{R}^n \to M$ with respect to which the connection is Euclidean. That is, $\varphi_* D_X Y = \nabla_{\varphi_* X} \varphi_* Y$ for all smooth vector fields $X, Y$ on $U$. If $\nabla$ is the Levi-Civita connection of a Riemannian metric, then the metric is flat if and only if the manifold is locally isometric to Euclidean space.

**Proof.** Because this is the assertion of a local fact, we may assume without loss of generality that $M = \mathbb{R}^n$. An important observation, which I may prove later when I find time, is that vanishing of the curvature tensor implies that the parallel transport map $P_\gamma$ is independent of the path $\gamma$ joining two points. (This, again, is a local statement about sufficiently small regions of $M$ but holds globally on $\mathbb{R}^n$.) This means that given vectors $u_1, \ldots, u_n \in T_0 \mathbb{R}^n$ we can find parallel vector fields $X_1, \ldots, X_n$ such that $X_i(0) = u_i$ for each $i$. Because these fields are parallel and the torsion tensor is zero, we conclude that $\{X_j, X_i\} = \nabla_{X_i} X_j - \nabla_{X_j} X_i = 0$. Therefore, these fields commute. Now problem 9 of homework set 7 implies that there exists a coordinate system (defined using the flows of the $X_i$) whose coordinate vector fields are the $X_i$. This means that in this coordinate system the coordinate vector fields are parallel. But this property precisely characterizes the Euclidean connection.

The torsion tensor is clearly antisymmetric:

$$T(u, v) = -T(v, u).$$

It is also clear from the definition of the curvature tensor that

$$R(u, v) w = -R(v, u) w.$$
**Proposition 10** (Bianchi’s identity). *The curvature tensor of a torsion-free connection satisfies*

\[ R(u, v)w + R(v, w)u + R(w, u)v = 0. \]

*Proof.* This is a straightforward consequence of the definition of \( R \) and the Jacobi identity. You’ll prove it in one of the exercises. \( \square \)

- **Submanifolds of \( \mathbb{R}^n \).** Let \( M \) be an (immersed) \( n \)-dimensional submanifold of \( \mathbb{R}^N \). The restriction to tangent spaces of \( M \) of the standard inner product of \( \mathbb{R}^N \) defines a Riemannian metric on \( M \). Let \( \Pi_p : \mathbb{R}^N \to T_p M \) denote the orthogonal projection.

There is also a connection \( \nabla \) on \( M \) induced from the standard connection \( D \) on \( \mathbb{R}^N \) given as follows. Let \( X \) be a smooth vector field on \( M \) and \( v \in T_p M \). Let \( \gamma(t) \) be a differentiable curve representing \( v \) so, by definition, \( \gamma(0) = p \) and \( \gamma'(0) = v \). Then we define

\[ \nabla_v X = \Pi_p \frac{DX}{dt} \bigg|_{t=0}. \]

It is not difficult to show that \( \nabla \) is indeed a connection on \( M \). Also note that a curve \( \gamma(t) \) in \( M \) is geodesic with respect to this connection if its Euclidean acceleration is always perpendicular to \( M \).

**Proposition 11.** *The connection \( \nabla \) on the submanifold \( M \subset \mathbb{R}^N \) induced from \( D \) as explained above is torsion-free and metric with respect to the Euclidean inner product restricted to tangent spaces to \( M \).*

*Proof.* We may assume without loss of generality that vector fields \( X, Y \) on \( M \) are defined and smooth on an open neighborhood of \( M \) in \( \mathbb{R}^N \). At any \( p \in M \) the Lie bracket \( [X, Y]_p \) only depends on the values of \( X, Y \) on \( M \). But regarded as a vector field in \( \mathbb{R}^N \) we have \( [X, Y] = DX Y - DY X \) because \( D \) is torsion-free. Projecting to \( T_p M \) under \( \Pi_p \) we obtain \( [X, Y] = \nabla_X Y - \nabla_Y X \). Thus \( \nabla \) is torsion-free. Similarly, for \( v \in T_p M, \)

\[ \langle D_v X - \nabla_v X, w \rangle_p = 0 \]

where I have used the fact that \( \langle D_v X - \nabla_v X, w \rangle_p = 0 \) for all \( v, w \in T_p M \). \( \square \)

- **The Levi-Civita connection.** We have just noted that the induced connection on submanifolds of Euclidean space is torsion-free and metric relative to the induced Riemannian metric. For a general Riemannian manifold it turns out that there exists a unique torsion-free metric connection. This is the default connection used in Riemannian geometry and is called the *Levi-Civita* connection of the Riemannian manifold.

**Proposition 12.** *On a Riemannian manifold \( M \) there exists a unique connection \( \nabla \) that is both metric and torsion-free, called the Levi-Civita connection. It is given explicitly by the following identity involving arbitrary smooth vector fields \( X, Y, Z \):

\[ \langle \nabla Y Z, X \rangle = \frac{1}{2} \left( \langle X, Y \rangle Z + \langle X, Z \rangle Y - \langle Y, Z \rangle X + \langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle - \langle [X, Z], Y \rangle \right) \]

*Proof.* If a torsion-free metric connection exists then

\[ X(Y, Z) + Y(X, Z) - Z(X, Y) = \langle \nabla X Y + \nabla Y X, Z \rangle + \langle \nabla X Z - \nabla Z X, Y \rangle + \langle \nabla Y Z - \nabla Z Y, X \rangle \]

\[ = 2 \langle \nabla X Y - \langle [X, Y], Z \rangle + \langle [X, Z], Y \rangle + \langle [Y, Z], X \rangle. \]

The stated identity follows by rearranging terms. \( \square \)
• **Computing curvatures using differential forms.** Let \((M, \langle \cdot, \cdot \rangle)\) be a Riemannian manifold of dimension \(n\) with Levi-Civita connection \(\nabla\). Let \(e_1, \ldots, e_n\) be smooth vector fields defined on an open set \(U \subset M\) and suppose that \(\{e_1(p), \ldots, e_n(p)\}\) constitutes an orthonormal basis of \(T_p M\) for each \(p \in M\). Let \(\theta_1, \ldots, \theta_n\) denote the dual one-forms, so that \(\theta_i(e_j) = \delta_{ij}\). We define the connection one-forms \(\omega_{ij}\) on \(U\) by

\[
\omega_{ij}(u) := \langle \nabla_u e_i, e_j \rangle.
\]

We also define on \(U\) the curvature 2-forms \(\Omega_{ij}\) by

\[
\Omega_{ij}(u, v) = \langle R(u, v)e_i, e_j \rangle.
\]

**Theorem 13** (Cartan’s structure equations). The connection forms and curvature forms satisfy: \(\omega_{ij} = -\omega_{ji}\) and \(\Omega_{ij} = -\Omega_{ji}\). Furthermore,

\[
d\theta_i = \sum_j \omega_{ij} \wedge \theta_j \quad (3)
\]

\[
d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} + \Omega_{ij} \quad (4)
\]

and the \(\omega_{ij}\) are uniquely determined by 3. In dimension 2, writing \(\omega = \omega_{12}\) and \(\Omega = \Omega_{12}\), then \(d\omega = \Omega\).

**Proof.** By the metric property of the connection

\[
0 = \langle X(e_i, e_j) \rangle = \langle \nabla_X e_i, e_j \rangle + \langle e_i, \nabla_X e_j \rangle.
\]

which implies \(\omega_{ij} = -\omega_{ji}\). From \(\langle R(\cdot, \cdot)e_i, e_j \rangle = -\langle R(\cdot, \cdot)e_j, e_i \rangle\) we obtain \(\Omega_{ij} = -\Omega_{ji}\).

Now recall the formula \(d\theta(X, Y) = X\theta(Y) - Y\theta(X) - \theta([X, Y])\) for the exterior derivative of a one-form \(\theta\). Applying this to \(\theta_i\) gives

\[
d\theta_i(X, Y) = X\theta_i(Y) - Y\theta_i(X) - \theta_i([X, Y])
\]

\[
= \langle \nabla_X e_i, Y \rangle - \langle Y e_i, X \rangle - \langle e_i, [X, Y] \rangle
\]

\[
= \langle \nabla_X e_i, Y \rangle + \langle e_i, \nabla_X Y \rangle - \langle \nabla_Y e_i, X \rangle - \langle e_i, \nabla_Y X \rangle - \langle e_i, [X, Y] \rangle
\]

\[
= \langle \nabla_X e_i, Y \rangle - \langle \nabla_Y e_i, X \rangle + \langle e_i, \nabla_X Y - \nabla_Y X - [X, Y] \rangle
\]

\[
= \langle \nabla_X e_i, Y \rangle - \langle \nabla_Y e_i, X \rangle
\]

where it was used that the torsion-tensor is 0. We also have

\[
\langle \nabla_X e_i, Y \rangle - \langle \nabla_Y e_i, X \rangle = \sum_j \left( \langle \nabla_X e_i, e_j \rangle \langle e_j, Y \rangle - \langle \nabla_Y e_i, e_j \rangle \langle e_j, X \rangle \right) = \sum_j (\omega_{ij}(X)\theta_j(Y) - \omega_{ij}(Y)\theta_j(X)).
\]
Therefore, \( d\theta_i = \sum_j \omega_{ij} \wedge \theta_j \). We now do a similar computation for \( d\omega_{ij} \).

\[
d\omega_{ij}(X,Y) = X\omega_{ij}(Y) - Y\omega_{ij}(X) - \omega_{ij}([X,Y])
\]

\[
= X \langle \nabla_Y e_i, e_j \rangle - Y \langle \nabla_X e_i, e_j \rangle - \langle \nabla_{[X,Y]} e_i, e_j \rangle
\]

\[
= \langle \nabla_Y e_i, \nabla_X e_j \rangle - \langle \nabla_X e_i, \nabla_Y e_j \rangle + \langle \nabla_X \nabla_Y X - \nabla_Y \nabla_X X, e_i, e_j \rangle
\]

\[
= \sum_k \langle \nabla_Y e_i, e_k \rangle \langle e_k, \nabla_X e_j \rangle - \sum_k \langle \nabla_X e_i, e_k \rangle \langle e_k, \nabla_Y e_j \rangle + \langle \nabla(X,Y) e_i, e_j \rangle
\]

\[
= \sum_k \omega_{ik}(Y) \omega_{kj}(X) - \omega_{ik}(X) \omega_{kj}(Y) \right) + \Omega_{ij}(X,Y)
\]

It remains to show that the \( \omega_{ij} \) are uniquely determined by the equations 3. If \( d\theta_i = \sum_j \sigma_{ij} \wedge \theta_j \) for one forms \( \sigma_{ij} \) then, setting \( \beta_{ij} = \omega_{ij} - \sigma_{ij} \), we have \( \sum_j \beta_{ij} \wedge \theta_j = 0 \). We need to show \( \beta_{ij} = 0 \). Expressing \( \beta_{ij} \) in the basis \( \{\theta_k\} \), \( \beta_{ij} = \sum_k a^k_{ij} \theta_k \), we see that \( a^k_{ij} = -a^i_{kj} \). On the other hand,

\[
0 = \sum_j \beta_{ij} \wedge \theta_j = \sum_j a^k_{ij} \theta_k \wedge \theta_j = \sum_{j<k} \left( a^k_{ij} - a^j_{ik} \right) \theta_k \wedge \theta_j.
\]

But the \( \theta_k \wedge \theta_j \) are linearly independent for \( k < j \) so the coefficients satisfy \( a^k_{ij} = a^j_{ik} \). From these relations for the \( a^k_{ij} \) we derive

\[
a^k_{ij} = a^k_{ji} = -a^j_{ki} = -a^i_{kj} = a^i_{jk} = a^j_{ik} = -a^k_{ji}.
\]

Therefore \( a^k_{ij} = 0 \), so \( \beta_{ij} = 0 \).

Let us write down the connection forms in terms of the frame of orthonormal covector fields.

**Proposition 14.** Let \( (M, \langle \cdot, \cdot \rangle) \) be a Riemannian manifold. Let \( e_1, \ldots, e_n \) be a local orthonormal frame and \( \theta_1, \ldots, \theta_n \) the dual frame as in Theorem 13. Define the functions \( h^k_{ij} \) by

\[
[e_i, e_j] = \sum_k h^k_{ij} e_k.
\]

Then the connection forms can be written as

\[
\omega_{ij} = \frac{1}{2} \left( i_{e_i} d\theta_j - i_{e_j} d\theta_i - \sum_k h^k_{ij} \theta_k \right).
\]

This is a straightforward application of the explicit form of the Levi-Civita connection given in Proposition 12.

- **Geometry of surfaces and Gaussian curvature.** Suppose \( e_1, e_2, \nu \) are a locally defined orthonormal frame on the surface \( M \subset \mathbb{R}^3 \), where \( e_1, e_2 \) are tangent to \( M \) and \( \nu \) is perpendicular. Note that \( D_u \nu \in T_p M \) at each \( p \in M \) and all \( u \in T_p M \) where \( D \) indicates the Euclidean connection in \( \mathbb{R}^3 \). This is because \( 0 = u(\nu \cdot \nu) = 2(D_u \nu) \cdot \nu \). The linear map \( S_p : T_p M \to T_p M \) defined by \( S_p u := -D_u \nu \) is called the **shape operator** at \( p \).

**Proposition 15** (The shape operator is symmetric). Let \( M \) be a surface in \( \mathbb{R}^3 \). Then \( \langle S_p u, v \rangle = \langle u, S_p v \rangle \) for all \( p \in M \) and \( u, v \in T_p M \).

**Proof.** If \( U, V \) are any local vector fields defined on a neighborhood of \( p \) so that \( U_p = u, V_p = v \), then

\[
0 = [U, V]_p \cdot v_p = (D_u V - D_v U) \cdot v_p = (-v \cdot D_u v + u \cdot D_v v) = v \cdot S_p u - u \cdot S_p v.
\]
This means that $S_p$ is symmetric.

It follows that $T_pM$ has an orthonormal basis of eigenvectors of $S_p$ associated to real eigenvalues $\lambda_1, \lambda_2$.

**Definition 16** (Principal curvatures; mean and Gauss curvatures). The eigenvalues $\lambda_1, \lambda_2$ of the shape operator $S_p$ are called the principal curvatures of the surface. The numbers

$$K(p) := \lambda_1 \lambda_2, \quad H(p) := \frac{\lambda_1 + \lambda_2}{2}$$

are called the Gauss curvature and the mean curvature, respectively.

The following theorem says that the surface curvature $K$, which was defined extrinsically (by how the normal vector changes in the ambient space $\mathbb{R}^3$) in fact only depends on intrinsic metric properties of the surface. By intrinsic I mean only information that can be gathered by a citizen of flatspace, able to measure lengths and angles on the surface, but who has no knowledge of the outside space.

**Theorem 17** (Gauss’s Theorem Egregium). Let $M$ be a surface in $\mathbb{R}^3$. Then at each $p \in M$ and for any orthonormal vectors $u, v \in T_pM$,

$$K(p) = \langle R_p(u, v)v, u \rangle$$

where $R$ is the curvature tensor associated to the induced Riemannian metric on $M$.

**Proof.** We begin from the observation that if $e_1, e_2$ constitute an orthonormal frame on an open set in $M$, then

$$D_{e_1} D_{e_2} e_2 - D_{e_2} D_{e_1} e_2 - D_{[e_1, e_2]} e_2 = 0.$$ 

In fact, this is the component $R_{w}^w(e_1, e_2)e_2$ of the Euclidean curvature tensor in $\mathbb{R}^3$. Let $\nabla$ be the Levi-Civita conniption on $M$. Then

$$D_{e_1} e_j = \nabla_{e_1} e_j + \langle D_{e_1} e_j, v \rangle v = \nabla_{e_1} e_j - \langle e_j, D_{e_1} v \rangle v = \nabla_{e_1} e_j + \langle Se_j, e_j \rangle v$$

where $S$ is the shape operator. From this we derive the second derivatives:

$$D_{e_1} D_{e_2} e_2 = D_{e_1} \{ \nabla_{e_2} e_2 + \langle Se_2, e_2 \rangle v \} = \nabla_{e_1} \nabla_{e_2} e_2 + \langle Se_1, \nabla_{e_2} e_2 \rangle v + e_1 \langle Se_2, e_2 \rangle v + \langle Se_2, e_2 \rangle D_{e_1} v.$$

Thus

$$D_{e_1} D_{e_2} e_2 = \nabla_{e_1} \nabla_{e_2} e_2 + \langle Se_1, \nabla_{e_2} e_2 \rangle v + e_1 \langle Se_2, e_2 \rangle v + \langle Se_2, e_2 \rangle D_{e_1} v.$$ 

Similarly:

$$D_{e_2} D_{e_1} e_2 = \nabla_{e_2} \nabla_{e_1} e_2 + \langle Se_2, \nabla_{e_1} e_2 \rangle v + e_2 \langle Se_1, e_2 \rangle v + \langle Se_1, e_2 \rangle D_{e_2} e_1.$$ 

$$D_{[e_1, e_2]} e_2 = \nabla_{[e_1, e_2]} e_2 + \langle S[e_1, e_2], e_2 \rangle v.$$

Then, if $\Pi_p$ denotes the orthogonal projection to $T_pM$,

$$0 = D_{e_1} D_{e_2} e_2 - D_{e_2} D_{e_1} e_2 - D_{[e_1, e_2]} e_2 = \nabla_{e_1} \nabla_{e_2} e_2 - \nabla_{e_2} \nabla_{e_1} e_2 - \nabla_{[e_1, e_2]} e_2 - \langle Se_2, e_2 \rangle \nabla_{e_1} e_2 + \langle Se_1, e_2 \rangle \nabla_{e_2} e_2 + Q$$

where $Q$ is a vector proportional to $v$. Thus, writing $S_{ij} = \langle Se_i, e_j \rangle$ and noting that $K(p)$ is the determinant of $S$, we conclude

$$0 = \langle R(e_1, e_2)e_2, e_1 \rangle - \langle S_{22} S_{11} - S_{12} S_{21} \rangle = \langle R(e_1, e_2)e_2, e_1 \rangle - K(p).$$
From the symmetries of \( R \) we have \( \langle R(e_1, e_2) e_2, e_1 \rangle = \langle R(u, v) v, u \rangle \) for an arbitrary orthonormal basis \( \{u, v\} \) of \( T_p M \). This concludes the proof.

Problems

Read all the problems given below and solve 4 of them.

1. **The Euclidean connection.** Prove that the Euclidean covariant derivative of vector fields satisfies:

   \( D_v (X + Y) = D_v X + D_v Y \)
   \( (c_1) \)
   \( D_v (f X) = (v f) X + f D_v X \)
   \( (c_2) \)
   \( D_{au + bv} X = a D_u X + b D_v X \)
   \( (c_3) \)
   \( v_p (X, Y) = \langle D_v X, Y \rangle + \langle X, D_v Y \rangle \)
   \( (c_4) \)
   \( D_u X - D_Y X = [X, Y] \)
   \( (c_5) \)
   \( D_v \{X, Y\} Z = D_{\{X, Y\}} Z. \)
   \( (c_6) \)

   Here \( \langle \cdot, \cdot \rangle \) is the standard inner product in \( \mathbb{R}^n \).

2. **Bianchi's identity.** Show that the identity

   \( R(u, v) w + R(v, w) u + R(w, u) v = 0 \)

   holds for a torsion-free connection \( \nabla \). [You'll need the Jacobi identity for the Lie bracket of vector fields. Note: Apply the definition of \( R \) to vector fields \( U, V, W \) which are equal to \( u, v, w \) at \( p \). The resulting identity will not depend on the choice of vector fields.]

3. **Other symmetries of \( R \).** Let now \( \nabla \) be the Levi-Civita connection of a Riemannian manifold \( M \) with Riemannian metric \( \langle \cdot, \cdot \rangle \). Prove the following identities:

   \( a \) \( \langle R(u, v) w, z \rangle = - \langle R(v, u) w, z \rangle \)
   \( (b) \)
   \( \langle R(u, v) w, z \rangle = - \langle R(u, z) v, w \rangle \)
   \( (c) \)
   \( \langle R(u, v) w, z \rangle = \langle R(w, z) u, v \rangle \)
   \( (c) \)

   (These identities are standard. You'll find the argument online if you get stuck.)

4. **Conformally Euclidean metrics.** Let \( \eta(x) = \rho^\alpha(x) \) be a positive smooth function on \( \mathbb{R}^n \) and define the metric

   \( \langle u, v \rangle_x = n(x)^2 u \cdot v. \)

   \( (a) \) Show that the Levi-Civita connection for this metric has the following expression on constant vector fields \( X, Y \):

   \( \nabla_X Y = \{X \rho\} Y + \{Y \rho\} X - X \cdot Y \text{ grad}_{\text{Eucl}} \rho \)

   where \( \text{grad}_{\text{Eucl}} \) indicates ordinary (Euclidean) gradient as opposed to the gradient in the new metric.

   \( (b) \) Show that a smooth path \( \gamma(t) = (x_1(t), \ldots, x_m(t)) \) is a geodesic for this metric if and only if it satisfies the system of differential equations

   \( \dot{x}_j + \sum_i \left[ 2 \dot{x}_i \dot{x}_j - | \dot{x} |^2 \delta_{ij} \right] \frac{\partial \rho}{\partial x_i} = 0, \quad j = 1, \ldots, m. \)  \( (5) \)
(c) Show that a vector field \( X(t) = \sum_j h_j(t) \partial/\partial x_j \) is parallel along a curve \( \gamma(t) = (x_1(t), \ldots, x_m(t)) \) if the functions \( h_j(t) \) satisfy the linear system of first order differential equations:
\[
\dot{h}_j + \sum_i \left\{ (h_j \dot{x}_i + h_i \dot{x}_j) - h \cdot \delta_{ij} \right\} \frac{\partial p}{\partial x_i} = 0, \quad j = 1, \ldots, m.
\]

5. **Conformally Euclidean metrics, continued.** Let us denote the standard Euclidean norm of a vector \( \nu \) by \( |\nu| \) so that \( \eta(\nu)|\nu| \) is the norm of \( \nu \) in the conformally Euclidean metric \( \langle \cdot, \cdot \rangle \) determined by \( \eta \).

(a) Show that \( \eta(\nu)|\nu| \) is constant in \( t \) if \( x = x(t) \) is a geodesic path with respect to \( \langle \cdot, \cdot \rangle \). (This is immediate if you start from \( \frac{d}{dt}(x, \dot{x}) \) and use the property that the connection is Riemannian.)

(b) Let us introduce the parameter \( \zeta(t) \) by \( d\zeta = \frac{1}{\eta(|x|)} \, dt \) and set \( z(\zeta) = x(t(\zeta)) \). Show that \( x(t) \) is a geodesic path for \( \langle \cdot, \cdot \rangle \) if and only if \( z(\zeta) \) satisfies the differential equation
\[
mz'' = -\nabla_{\nu} U
\]
where \( m = (\eta(|x|))^{-1/2} \) is a constant of motion by part (a) and \( U = E - \frac{1}{2} \eta^2 \) for an arbitrary constant \( E \). Note that this is Newton's equation for a mechanical particle with mass \( m \) moving relative to time parameter \( \zeta \) under the influence of a potential \( U = E - \frac{1}{2} \eta^2 \). Because \( |z'| = |x| \eta^2 = m^{-1/2} \eta \), it follows that
\[
E = \frac{1}{2} m |z'|^2 + U.
\]

(c) Let us now assume that \( \eta \) is only a function of the distance \( r = |x| = |z| \) from \( z \) to the origin. Thus
\[
\eta(r) = \sqrt{2(E-U(r))}
\]
for a central potential \( U(r) \). Show that the equation of geodesic reduces to
\[
z'' = f(|z|)z.
\]
where \( f(r) = -\frac{U'(r)}{m} \). Also show that the solution path of the initial value problem lies in the plane spanned by the initial conditions \( z(0), z'(0) \). (This is immediate from existence and uniqueness of solutions.) Therefore, in the spherically symmetric case the problem is essentially 2-dimensional.

6. **Curvature is an invariant of the Riemannian geometry.** Let \( M \) and \( N \) be two Riemannian manifolds with Riemannian metrics \( \langle \cdot, \cdot \rangle_m \) and \( \langle \cdot, \cdot \rangle_n \). Suppose \( M \) and \( N \) are **isometric**; that is, there exists a diffeomorphism \( F : M \to N \) such that
\[
\langle dF_p u, dF_p v \rangle_n = \langle u, v \rangle_m
\]
for all \( u, v \in T_p M \). Show that the pull-back under \( F \) of the curvature tensor \( R^m \) of \( N \) equals the curvature tensor \( R^n \) of \( M \).

7. **Conformal transformations.** A conformal mapping of a region (i.e., a connected open set) \( U \subset \mathbb{R}^2 \) is a diffeomorphism \( F : U \to V := F(U) \) that preserves Euclidean angles. Thus, by definition, \( dF_z u = \eta(x) u \) for some function \( \eta : U \to (0, \infty) \). Continuous functions of one complex variable on regions in \( \mathbb{C} \), regarded as real maps, are conformal. For example, \( F(x, y) = (x^2 - y^2, 2xy) \) is a conformal mapping coming from the complex variable function \( f(z) = z^2, \; z = x + iy \). Define on \( U \) the Riemannian metric
\[
\langle u, v \rangle_x = \eta^2(x) u \cdot v.
\]
Show that this conformally Euclidean metric is flat. (This is immediate given the previous problem.) Also show the converse: If a conformally Euclidean metric is flat, then it is the pull-back of the Euclidean metric under some mapping $F$ (which is then conformal).

8. Geodesics and curvature of the 2-sphere bis. Show:

(a) Geodesics of $S^2$ are great circles; that is, they are the circles of intersection of $S^2$ with a plane in $\mathbb{R}^3$ passing through the origin. (To each point $x \in S^2$ and tangent vector $v \in T_xS^2$ there exists a unique great circle containing $x$ and tangent to $v$. If you show that great circles parametrized so as to have constant speed are geodesics, then by uniqueness the geodesic with initial condition $(x,v)$ is this great circle.)

(b) The Gauss curvature of $S^2(r) = \{ x \in \mathbb{R}^3 : |x| = r \}$ is constant equal to $K = 1/r^2$. (I suggest here using a different method than in the previous problem: Use spherical coordinates to obtain an orthonormal frame on an open subset of the sphere. Note: because the sphere admits a transitive group of isometries given by the rotation group in $\mathbb{R}^3$, all points of $S^2$ have the same Gauss curvature. So it suffices to compute $K$ at a single point.)

(c) (Alternative to previous item.) Show that $S^2(r)$ has curvature $1/r^2$ by first noting that the shape operator is scalar, $S = (1/r)I$, then using Gauss’s Theorem Egregium.

9. The hyperbolic plane. Let $\mathbb{H}^2$ be the upper half-plane, that is

$$\mathbb{H}^2 = \{ (x, y) \in \mathbb{R}^2 : y > 0 \}.$$

Consider in $\mathbb{H}^2$ the following conformally Euclidean inner product: if $p = (x, y)$ and $u, v \in T_p\mathbb{H}^2$

$$\langle u, v \rangle_p = \frac{u \cdot v}{y^2}.$$

Prove that the Gaussian curvature is constant equal to $K = -1$. With this choice of metric $\mathbb{H}^2$ is called the hyperbolic plane.

10. Curvature of a graph. Let $M = \{(x, x, f(x, y)) : (x, y) \in \mathbb{R}^2\}$ where $f(x, y) = (ax^2 + by^2)/2$ for real constants $a, b$. Find the Gauss curvature $K$ at the point $(0,0,0)$. 

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