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General instructions: This exam has 16 questions, each worth the same amount. Check that no pages are missing and notify your proctor if you detect any problems with your copy of the exam. Mark your ID number on the six blank lines on the top of your answer card, using one line for each digit. Print your name on the top of the card. Choose the answer that is closest to the solution and mark your answer card with a PENCIL by shading in the correct box. You may use a 3×5 card with notes and any calculator that does not have graphing functions. GOOD LUCK!

1. Solve the initial value problem

\[ \frac{d^2 r}{dt^2} = -2k \]

with initial conditions

\[ r(0) = 20k \quad \text{and} \quad \frac{dr}{dt} \bigg|_{t=0} = i + 3j. \]

(A) \( r(t) = 2ti + tj + (10 - t^2)k \)
(B) \( r(t) = ti - tj + (20 + t^2)k \)
(C) \( r(t) = 3ti - 2tj + (20 + t^2)k \)
(D) \( r(t) = 3ti - tj + (40 - t^2)k \)
(E) \( r(t) = -ti - j + (20 - t^2)k \)
(F) \( r(t) = ti + 3tj + (20 - t^2)k \)
(G) \( r(t) = 4ti - 2tj + (t^2 - 10)k \)
(H) \( r(t) = ti - 3tj + (t^2 - 20)k \)
(I) \( r(t) = ti + (t^2 - 20)k \)
(J) \( (t^2 - 20)k \)
Solution: Integrating $-2\mathbf{k}$ once from 0 to $t$ gives $\mathbf{r}'(t) = \mathbf{r}'(0) - 2t\mathbf{k}$ and integrating a second time gives

$$\mathbf{r}(t) = \mathbf{r}(0) + \mathbf{r}'(0)t - t^2\mathbf{k}.$$ 

Using the initial conditions:

$$\mathbf{r}(t) = 20\mathbf{k} + (\mathbf{i} + 3\mathbf{j})t - t^2\mathbf{k} = t\mathbf{i} + 3t\mathbf{j} + (20 - t^2)\mathbf{k}.$$
2. Decide whether each of the following statements is always true (T), or sometimes false or simply meaningless (F):

   a. \((u \times v) \cdot w = u \cdot (v \times w)\)
   b. \((u \times v) \cdot w = u \times (v \cdot w)\)
   c. \((u \cdot v) \cdot w = u \cdot (v \cdot w)\)
   d. If \(u \times v = u \times w\) and \(u \neq 0\), then \(v = w\)
   e. \((u \times v) \cdot v = 0\)

These are, respectively:

   (A) F, F, F, F, F
   (B) F, F, F, T, F
   (C) T, F, T, T, F
   (D) T, T, T, F, F
   (E) T, F, T, F, T
   (F) F, T, F, T, F
   (G) T, T, T, F, F,
   (H) T, F, T, T, T
   (I) T, F, F, F, T
   (J) F, F, F, T, T
Solution: a. is always true as discussed in class; b. and c. are meaningless; d. is not always true: take, for example, $\mathbf{v} = \mathbf{u} = \mathbf{i}$, and $\mathbf{w} = \mathbf{0}$; d. is always true as the cross product is always perpendicular to both factors.
3. Find a parametric equation for the tangent line to the curve
\[ \mathbf{r}(t) = \langle \sin t, t^2 - \cos t, e^t \rangle \]
at the point with parameter value \( t = 0 \).

(A) \( x = 2t, \ y = 0, \ z = 1 - t \)
(B) \( x = t, \ y = 1, \ z = 1 - t \)
(C) \( x = t, \ y = -1, \ z = 1 + t \)
(D) \( x = t, \ y = 2, \ z = 1 + t \)
(E) \( x = t, \ y = -1, \ z = 1 + 2t \)
(F) \( x = t, \ y = -1, \ z = 1 + 3t \)
(G) \( x = t, \ y = -t, \ z = 1 + t \)
(H) \( x = t, \ y = -1 - t, \ z = 1 + t \)
(I) \( x = t, \ y = -1 - 2t, \ z = 1 + t \)
(J) \( x = 2t, \ y = -1 - t, \ z = 1 + t \)
Solution:
We have $\mathbf{r}(0) = \langle 0, -1, 1 \rangle$ and $\mathbf{r}'(0) = \langle 1, 0, 1 \rangle$. Therefore, the vector equation of the tangent line is

$$s(t) = \mathbf{r}(0) + \mathbf{r}'(0)t = \langle 0, -1, 1 \rangle + t\langle 1, 0, 1 \rangle = \langle t, -1, 1 + t \rangle.$$
4. Find the point on the curve

\[ \mathbf{r}(t) = (5 \sin t, 5 \cos t, 12t) \]

at a distance $26\pi$ units along the curve from the point $(0, 5, 0)$ in the direction of increasing arc length.

\[\begin{array}{ll}
(A) & (0, 5, 24\pi) \\
(B) & (0, 5, 12\pi) \\
(C) & (0, 5, 6\pi) \\
(D) & (0, 5, 4\pi) \\
(E) & (0, 5, 3\pi) \\
(F) & (5, 0, 24\pi) \\
(G) & (5, 0, 12\pi) \\
(H) & (5, 0, 6\pi) \\
(I) & (5, 0, 4\pi) \\
(J) & (5, 0, 3\pi) \\
\end{array}\]
Solution: Notice that (0, 5, 0) = r(0). So we take the initial parameter to be \( t = 0 \). Also \( r'(t) = \langle 5 \cos t, -5 \sin t, 12 \rangle \), so
\[
|r'(t)| = \sqrt{25 + 144} = 13.
\]
Thus the arc length parameter is
\[
s(t) = \int_0^t |r'(\tau)| \, d\tau = \int_0^t 13 \, d\tau = 13t.
\]
Now \( s(t) = 26\pi \) implies \( t = 2\pi \) and the position vector of the point we want is
\[
r(2\pi) = \langle 0, 5, 24\pi \rangle.
\]
5. Find the curvature $\kappa$ for the curve

$$\mathbf{r}(t) = (e^t \cos t, e^t \sin t, 2)$$

at the point with parameter $t = 1$.  

(A) $e\sqrt{2}$  
(B) $\sqrt{2}$  
(C) $2\sqrt{2}$  
(D) $1/\sqrt{2}$  
(E) $1/(2\sqrt{2})$  
(F) $1/(e^2\sqrt{2})$  
(G) $1/(e\sqrt{2})$  
(H) $e/\sqrt{2}$  
(I) $e^2/\sqrt{2}$  
(J) $e/(2\sqrt{2})$
Solution:

The first derivative of $\mathbf{r}(t)$ is

$$\mathbf{v}(t) = \langle e^t(\cos t - \sin t), e^t(\sin t + \cos t), 0 \rangle$$

and its magnitude is

$$|\mathbf{v}(t)| = e^t \sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2} = e^t \sqrt{2}.$$

Thus the unit tangent vector is

$$\mathbf{T}(t) = \frac{1}{\sqrt{2}} \langle \cos t - \sin t, \sin t + \cos t, 0 \rangle.$$

Now,

$$\frac{d\mathbf{T}}{ds} = \frac{1}{|\mathbf{v}(t)|} \frac{d\mathbf{T}}{dt} = \frac{1}{e^t \sqrt{2}} \left( \frac{1}{\sqrt{2}} \langle -\sin t - \cos t, \cos t - \sin t, 0 \rangle \right).$$

The vector in parenthesis on the right-hand side has unit length and the multiplicative term in front of it is positive. So we can write

$$\frac{d\mathbf{T}}{ds} = \frac{1}{e^t \sqrt{2}} \mathbf{N}.$$

Therefore,

$$\kappa(t) = \frac{1}{e^t \sqrt{2}}$$

which is equal to $1/(e \sqrt{2})$ when $t = 1.$
6. Match the following surface equations:

a. \( x^2 + y^2 + 4z^2 = 10 \)
b. \( x^2 + 4z^2 = y^2 \)
c. \( x = z^2 - y^2 \)
d. \( x^2 + 2z^2 = 8 \)

with the surface descriptions:

1. hyperbolic paraboloid
2. cylinder with an elliptic cross-section
3. elliptical cone
4. ellipsoid

(A) \( a \rightarrow 1, b \rightarrow 2, c \rightarrow 3, d \rightarrow 4 \)
(B) \( a \rightarrow 4, b \rightarrow 3, c \rightarrow 2, d \rightarrow 1 \)
(C) \( a \rightarrow 1, b \rightarrow 4, c \rightarrow 3, d \rightarrow 2 \)
(D) \( a \rightarrow 2, b \rightarrow 3, c \rightarrow 4, d \rightarrow 1 \)
(E) \( a \rightarrow 3, b \rightarrow 2, c \rightarrow 1, d \rightarrow 4 \)
(F) \( a \rightarrow 4, b \rightarrow 3, c \rightarrow 1, d \rightarrow 2 \)
(G) \( a \rightarrow 4, b \rightarrow 1, c \rightarrow 3, d \rightarrow 2 \)
(H) \( a \rightarrow 4, b \rightarrow 2, c \rightarrow 3, d \rightarrow 1 \)
(I) \( a \rightarrow 3, b \rightarrow 1, c \rightarrow 2, d \rightarrow 4 \)
(J) \( a \rightarrow 4, b \rightarrow 1, c \rightarrow 2, d \rightarrow 3 \)
Solution: This is readily deduced from the general form of these equations.
7. Find an equation for the circle of curvature of the curve \( r(t) = (t, \sin t) \) at the point \((\pi/2, 1)\).

\[(A) \quad (x - \frac{\pi}{2})^2 + (y + 5)^2 = 1
(B) \quad (x - \frac{\pi}{2})^2 + (y + 4)^2 = 1
(C) \quad (x - \frac{\pi}{2})^2 + (y + 3)^2 = 1
(D) \quad (x - \frac{\pi}{2})^2 + (y + 2)^2 = 1
(E) \quad (x - \frac{\pi}{2})^2 + (y + 1)^2 = 1
(F) \quad (x - \frac{\pi}{2})^2 + y^2 = 5
(G) \quad (x - \frac{\pi}{2})^2 + y^2 = 4
(H) \quad (x - \frac{\pi}{2})^2 + y^2 = 3
(I) \quad (x - \frac{\pi}{2})^2 + y^2 = 2
(J) \quad (x - \frac{\pi}{2})^2 + y^2 = 1\]
**Solution:** The point $(\pi/2, 1)$ corresponds to the parameter, $t = \pi/2$. The velocity of the curve is $v(t) = \langle 1, \cos t \rangle$, and the acceleration is $a(t) = \langle 0, -\sin t \rangle$. At $t = \pi/2$ we have

$$v = \langle 1, 0 \rangle = \mathbf{i}, \quad a = \langle 0, -1 \rangle = -\mathbf{j}.$$ We can find $\kappa$ by the formula

$$\kappa = \frac{|v \times a|}{|v|^3} = 1.$$ Thus the circle of curvature has radius $R = 1$ and center

$$\overrightarrow{OC} = \langle \pi/2, 1 \rangle - \mathbf{j} = \langle \pi/2, 0 \rangle$$ Therefore, the equation of the circle is

$$\left(x - \frac{\pi}{2}\right)^2 + y^2 = 1.$$
8. Find the volume of the segment cut from the paraboloid \( z = x^2 + y^2 \) by the plane \( z = 1 \).

(In other words, find the volume of the region defined by \( x^2 + y^2 \leq z \leq 1 \).)

(A) \( \frac{\pi}{3} \)

(B) \( \frac{\pi}{2} \)

(C) \( \pi \)

(D) \( \frac{3\pi}{2} \)

(E) \( 2\pi \)

(F) \( \frac{\pi^2}{2} \)

(G) \( \frac{\pi^2}{3} \)

(H) \( \frac{3\pi^2}{2} \)

(I) \( \frac{2\pi^2}{3} \)

(J) \( 1 \)
Solution:

The cross-section of the region at level $z = r^2$ is a disc of radius $r$, whose area is $A(z) = \pi z$. Thus the volume of the region is

$$V = \int_{0}^{1} \pi z \, dz = \frac{\pi}{2}.$$
9. Find the velocity, speed, and acceleration at time $t = \pi$ of a particle in space whose position vector is $\mathbf{r}(t) = \langle 2 \cos t, 3 \sin t, 4t \rangle$.

(A) $\mathbf{v} = \langle 0, 0, 1 \rangle$, $v = 1$, $\mathbf{a} = \langle 2, 0, 0 \rangle$
(B) $\mathbf{v} = \langle 0, 5, -5 \rangle$, $v = 5\sqrt{2}$, $\mathbf{a} = \langle 2, 0, 0 \rangle$
(C) $\mathbf{v} = \langle 0, -3, 4 \rangle$, $v = 5$, $\mathbf{a} = \langle 0, 1, 0 \rangle$
(D) $\mathbf{v} = \langle 0, 4, 3 \rangle$, $v = 5$, $\mathbf{a} = \langle 1, 0, 0 \rangle$
(E) $\mathbf{v} = \langle 0, 4, -3 \rangle$, $v = 5$, $\mathbf{a} = \langle 2, 0, 0 \rangle$
(F) $\mathbf{v} = \langle 0, -3, 4 \rangle$, $v = 5$, $\mathbf{a} = \langle 1, 0, 0 \rangle$
(G) $\mathbf{v} = \langle 0, -3, 4 \rangle$, $v = 5$, $\mathbf{a} = \langle 0, 2, 0 \rangle$
(H) $\mathbf{v} = \langle 0, -1, 1 \rangle$, $v = \sqrt{2}$, $\mathbf{a} = \langle 2, 0, 0 \rangle$
(I) $\mathbf{v} = \langle 0, -4, 4 \rangle$, $v = 4\sqrt{2}$, $\mathbf{a} = \langle 1, 0, 0 \rangle$
(J) $\mathbf{v} = \langle 0, -3, 4 \rangle$, $v = 5$, $\mathbf{a} = \langle 2, 0, 0 \rangle$
Solution:

The velocity of the particle is

$$\mathbf{r}'(\pi) = \langle -2 \sin t, 3 \cos t, 4 \rangle|_{t=\pi} = \langle 0, -3, 4 \rangle,$$

the acceleration is

$$\mathbf{r}''(\pi) = \langle -2 \cos t, -3 \sin t, 0 \rangle|_{t=\pi} = \langle 2, 0, 0 \rangle,$$

the speed is

$$|\mathbf{r}'(\pi)| = \sqrt{9 + 16} = 5.$$
10. A projectile is fired with an initial speed of 100 m/sec at an angle of elevation of 45°. How high overhead will be projectile be 1 km downrange? (Use the approximation \( g = 9.8 \text{ m/sec}^2 \).) 

(A) 5 m  
(B) 9.8 m  
(C) 10 m  
(D) 20 m  
(E) 25 m  
(F) 30 m  
(G) 40 m  
(H) 50 m  
(I) 100 m  
(J) 150 m
Solution:
The parametric equation of the projectile in vector form is

\[ \mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0 t - \frac{1}{2} gt^2 \mathbf{j} \]

where \( \mathbf{r}_0 = \langle 0, 0 \rangle \) and \( \mathbf{v}_0 = 100 \langle \cos 45^\circ, \sin 45^\circ \rangle \). Therefore,

\[ \langle x, y \rangle = \langle \frac{100}{\sqrt{2}} t, \frac{100}{\sqrt{2}} t - \frac{9.8}{2} t^2 \rangle. \]

When \( x = 1000 \) meters downrange, \( t = 10\sqrt{2} \), and the height is at that same time is

\[ y = \frac{100}{\sqrt{2}} 10\sqrt{2} - \frac{9.8}{2} (10\sqrt{2})^2 = 1000 - 980 = 20. \]

So the height is 20 meters.
11. Find the tangential and normal scalar components of the acceleration of a particle that moves according to the function

\[ \mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, 3t \rangle. \]

\[ a_T = 0, \ a_N = 2t \]
\[ a_T = 0, \ a_N = t \]
\[ a_T = 1, \ a_N = 4 \]
\[ a_T = 3, \ a_N = 2 \]
\[ a_T = 2, \ a_N = 2 \]
\[ a_T = 0, \ a_N = 4 \]
\[ a_T = 0, \ a_N = 3 \]
\[ a_T = 0, \ a_N = 2 \]
\[ a_T = 1, \ a_N = 1 \]
\[ a_T = -1, \ a_N = 2 \]
Solution:
The velocity is
\[ \mathbf{v}(t) = (-2 \sin t, 2 \cos t, 3) \]
and the acceleration is
\[ \mathbf{a}(t) = (-2 \cos t, -2 \sin t, 0). \]
The speed is
\[ |\mathbf{v}(t)| = \sqrt{4 + 9} = \sqrt{13} \]
so the tangential component of the acceleration is
\[ a_T = \frac{d|\mathbf{v}(t)|}{dt} = 0. \]
The normal component is
\[ a_N = \sqrt{||\mathbf{a}||^2 - a_T^2} = \sqrt{4-0} = 2. \]
12. Find the distance from the point \((2, -3, 4)\) to the plane \(x + 2y + 2z = 13\).

(A) \(-1\)
(B) 0
(C) 1
(D) 2
(E) 3
(F) 4
(G) 5
(H) 6
(I) 7
(J) 8
Solution:
The point $Q = (13, 0, 0)$ is easily seen to lie on the plane, and $\mathbf{n} = \frac{1}{3}(1, 2, 2)$ is a unit perpendicular vector to the plane. The distance from $P = (2, -3, 4)$ to the plane is

$$D = \left| \overrightarrow{QP} \cdot \mathbf{n} \right| = \left| (-11, -3, 4) \cdot \frac{1}{3}(1, 2, 2) \right| = \left| \frac{-11 - 6 + 8}{3} \right| = 3.$$
13. Find the unit binormal vector $\mathbf{B}$ and the torsion function $\tau$ of the curve

$$\mathbf{r}(t) = \langle \cos t, \sin t, -1 \rangle$$

at the point with parameter $t = \pi/4$.

(A) $\mathbf{B} = \mathbf{i}, \quad \tau = 0$
(B) $\mathbf{B} = -\mathbf{k}, \quad \tau = -1$
(C) $\mathbf{B} = \mathbf{k}, \quad \tau = 1$
(D) $\mathbf{B} = -\mathbf{k}, \quad \tau = 0$
(E) $\mathbf{B} = \mathbf{k}, \quad \tau = 0$
(F) $\mathbf{B} = \sin t\mathbf{i} + \cos t\mathbf{j}, \quad \tau = 1/t$
(G) $\mathbf{B} = \cos t\mathbf{i} + \sin t\mathbf{j}, \quad \tau = 0$
(H) $\mathbf{B} = \sin t\mathbf{i} - \cos t\mathbf{j}, \quad \tau = t$
(I) $\mathbf{B} = \sin t\mathbf{j} + \cos tk, \quad \tau = t$
(J) $\mathbf{B} = \sin t\mathbf{i} + \cos tk, \quad \tau = 0$
Solution:

This is a plane curve that lies on the plane $z = -1$. Therefore, $B = \pm k$ and $\tau = 0$. An easy inspection gives $B = k$. 
14. Find the point in which the line

\[ x = 1 - t, \quad y = 3t, \quad z = 1 + t \]

meets the plane

\[ 2x - y + 3z = 6. \]

(A) \((-\frac{1}{2}, \frac{1}{2}, \frac{3}{2})\)
(B) \((\frac{1}{2}, -\frac{1}{2}, \frac{3}{2})\)
(C) \((0, 0, 2)\)
(D) \((3, 0, 0)\)
(E) \((1, -1, 1)\)
(F) \((2, -2, 0)\)
(G) \((-\frac{3}{2}, \frac{3}{2}, -\frac{1}{2})\)
(H) \((\frac{3}{2}, -\frac{3}{2}, \frac{1}{2})\)
(I) \((\frac{3}{2}, -\frac{3}{2}, \frac{1}{2})\)
(J) \((\frac{3}{2}, -\frac{3}{2}, \frac{1}{2})\)
Solution:
The parameter $t$ for the point we want satisfies

$$2(1 - t) - 3t + 3(1 + t) = 6.$$ 

This gives $t = -1/2$. Substituting into the parametric equation for the line gives

$$x = \frac{3}{2}, y = -\frac{3}{2}, z = \frac{1}{2}.$$
15. Match the function expressions

I. \( f(x, y) = \cos^{-1}(y - x^2) \)
II. \( f(x, y) = \sqrt{(x^2 - 4)(y^2 - 9)} \)
III. \( f(x, y) = \sqrt{y - x - 2} \)

with their domains among the sketched regions.

\[ \begin{array}{cccc}
\text{a} & \text{b} & \text{c} & \text{d} \\
\text{e} & \text{f} & & \\
\end{array} \]

(A) \( I \rightarrow f, \ II \rightarrow d, \ III \rightarrow a \)
(B) \( I \rightarrow a, \ II \rightarrow d, \ III \rightarrow c \)
(C) \( I \rightarrow e, \ II \rightarrow d, \ III \rightarrow b \)
(D) \( I \rightarrow f, \ II \rightarrow c, \ III \rightarrow a \)
(E) \( I \rightarrow a, \ II \rightarrow d, \ III \rightarrow f \)
(F) \( I \rightarrow d, \ II \rightarrow e, \ III \rightarrow b \)
(G) \( I \rightarrow b, \ II \rightarrow d, \ III \rightarrow c \)
(H) \( I \rightarrow f, \ II \rightarrow a, \ III \rightarrow e \)
(I) \( I \rightarrow d, \ II \rightarrow c, \ III \rightarrow a \)
(J) \( I \rightarrow b, \ II \rightarrow c, \ III \rightarrow f \)
Solution:
The domain of $\cos^{-1}(y - x^2)$ is determined by the inequalities $-1 \leq y - x^2 \leq 1$, which are equivalent to $x^2 - 1 \leq y \leq x^2 + 1$. This is the region between two parabolas, f.

The domain of $\sqrt{(x^2 - 4)(x^2 - 9)}$ consists of points such that

$$x^2 \geq 4 \quad \text{and} \quad y^2 \geq 9,$$

or

$$x^2 \leq 4 \quad \text{and} \quad y^2 \leq 9,$$

which corresponds to d.

The domain of $\sqrt{y - x - 2}$ is the set of points such that $y - x - 2 \geq 0$, or $y \geq 2 + x$. This region is shown in a.
16. The limits

I.
\[ \lim_{(x,y) \to (1,1)} \frac{xy - y - 2x + 2}{x - 1} \]

II.
\[ \lim_{(x,y) \to (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} \]

are, respectively:

(A) 0 and 0  
(B) $-1/2$ and $1/2$  
(C) 2 and $-1$  
(D) $-2$ and 1  
(E) 1 and 0  
(F) $-1$ and 0  
(G) 0 and 1  
(H) $-1$ and 1  
(I) 1 and $-1$  
(J) $-1$ and 2
Solution:

For the first limit notice that the numerator is \((x - 1)(y - 2)\) so that for \(x \neq 1\) the quotient is equal to \(y - 2\). This function is continuous, therefore the limit is \(1 - 2 = -1\).

For the second limit, notice that \(\lim_{(x,y) \to (0,0)} (x^2 + y^2) = 0\) and recall that the limit of \(\frac{\sin \theta}{\theta}\) as \(\theta \to 0\) is 1. Therefore, the second limit is 1.

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