The test contains 10 questions, each of equal value. Whenever possible, answers should be written using exact numbers. For example: write $\frac{2}{3}$ instead of 0.67, $\pi$ instead of 3.1415, $e^2$ instead of 7.4, etc.

Make sure to complete the test in one sitting, within 2 hours. Anything you write beyond the time limit should be clearly marked, say by changing the color of pen or pencil.

I will be available in my office all days till the due date, from 12 noon to 3 PM at least. I can also be available at other times by request.

Good luck!
1. Evaluate the integral \( \iint_{V} \mathbf{V} \cdot \mathbf{n} \, d\sigma \) over the surface of a sphere with center at the origin and radius 3, for
\[
\mathbf{V} = x(\cos^2 y)i + xzj + z(\sin^2 y)k.
\]
2. Evaluate the integral \( \oint_{C} \mathbf{V} \cdot d\mathbf{r} \) around the circle \((x - 2)^2 + (y - 3)^2 = 9, z = 0\), where
\[
\mathbf{V} = (x^2 + yz^2)i + (2x - y^3)j.
\]
3. Find all the eigenvalues and corresponding eigenvector of the matrix
\[
\begin{pmatrix}
2 & 0 & 2 \\
0 & 3 & 0 \\
2 & 0 & 2
\end{pmatrix}.
\]
4. Diagonalize the matrix
\[
M = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.
\]
In other words, find an orthogonal matrix \(C\) and a diagonal matrix \(D\) such that \(M = CDC^{-1}\).
5. Let \(f(x) = x\) on the interval \(-1 < x < 1\). Sketch the corresponding periodic function of period 2 and expand it in a complex Fourier series of period 2.
6. We already know the following Legendre polynomials:
\[
P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2 - 1).
\]
   (a) Using Rodrigues’ formula, find \(P_3(x)\).
   (b) Using a recursion relation, find \(P_4(x)\).
7. Expand the function \(f(x) = |x|, -1 < x < 1\), in Legendre series. (Find the series up to terms of degree 4.)
8. A bar 20 cm long with insulated sides is initially at 50°. Starting at \(t = 0\), the ends are held at 0°. Find the temperature distribution in the bar at time \(t\).
9. A string of length 2 has 0 initial velocity and displacement \(y_0(x)\) given by
\[
V(x) = \frac{1}{2} - \frac{1}{2}|x - 1|
\]
for \(0 \leq x \leq 2\). Its ends are fixed for all \(t\). Find the displacement as a function of \(x\) and \(t\).
10. Find the steady-state temperature distribution inside a sphere of radius 1, when the surface temperatures are given by \(h(\theta) = \cos \theta - 3\sin^2 \theta\).