Orthogonal 2 × 2 Matrices

Suppose $P$ is an $n \times n$ orthogonal matrix. Since $P$ is square and $P^{-1} = P^T$, we have
\[ 1 = \det(PP^{-1}) = \det(P^T) = (\det P)(\det P^T) = (\det P)^2, \]
so $\det P = \pm 1$.

**Theorem** Suppose $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is orthogonal. If

- if $\det P = 1$, then the mapping $x \to Px$ is a rotation
- if $\det P = -1$, then the mapping $x \to Px$ is a reflection across a line $L$ through the origin ($L$ is one of the eigenspaces of $P$)

**Proof** Since $P$ is orthogonal, the columns are orthonormal vectors in $\mathbb{R}^2$, so
\[ a^2 + c^2 = 1 = b^2 + d^2. \]
Therefore $\begin{bmatrix} a \\ c \end{bmatrix}$ and $\begin{bmatrix} b \\ d \end{bmatrix}$ are on the unit circle. Write $\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$, where $\theta$ is the angle between the positive $x$-axis and $\begin{bmatrix} a \\ c \end{bmatrix}$. Because $\begin{bmatrix} b \\ d \end{bmatrix}$ and $\begin{bmatrix} a \\ c \end{bmatrix}$ are orthogonal, $\begin{bmatrix} b \\ d \end{bmatrix}$ must be in one of the two positions $Q$ or $Q'$ (see picture); therefore
\[ \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} \cos (\theta + \frac{\pi}{2}) \\ \sin (\theta + \frac{\pi}{2}) \end{bmatrix} \text{ or } \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} \cos (\theta + \frac{3\pi}{2}) \\ \sin (\theta + \frac{3\pi}{2}) \end{bmatrix}. \]

So $P = \begin{bmatrix} \cos \theta & \cos \mu \\ \sin \theta & \sin \mu \end{bmatrix}$ where either $\mu = \theta + \frac{\pi}{2}$ or $\mu = \theta + \frac{3\pi}{2}$.

i) If $\mu = \theta + \frac{\pi}{2}$, then
\[ \cos \mu = \cos \theta \cos \frac{\pi}{2} - \sin \theta \sin \frac{\pi}{2} = -\sin \theta, \]
\[ \sin \mu = \sin \theta \cos \frac{\pi}{2} + \cos \theta \sin \frac{\pi}{2} = \cos \theta, \]
so $P = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$, which is a rotation matrix.
ii) If \( \theta = \theta + \frac{3\pi}{2} \), the same trig identities give 

\[
\begin{bmatrix}
\cos \theta & \sin \theta \\
\sin \theta & -\cos \theta
\end{bmatrix}
\]

In this case, 

\[
\det
\begin{bmatrix}
\cos \theta - \lambda & \sin \theta \\
\sin \theta & -\cos \theta - \lambda
\end{bmatrix} = \lambda^2 - 1,
\]

so \( P \) has eigenvalues \( \lambda_1 = 1 \) and \( \lambda_2 = -1 \). Pick corresponding eigenvectors \( b_1 \) and \( b_2 \).

Then \( b_1 \cdot b_2 = \frac{Pb_1 \cdot Pb_2}{P} \) (because \( P \) is orthogonal) 

\[
= (1) b_1 \cdot (-1)b_2
\]

\[= - b_1 \cdot b_2, \text{ so } b_1 \cdot b_2 = 0.\]

Since \( b_1 \) and \( b_2 \) are orthogonal, they are linearly independent. (Another reason: they are linearly independent because they are eigenvectors corresponding to different eigenvectors.)

Therefore \( \{b_1, b_2\} \) is an eigenvector basis for \( \mathbb{R}^2 \). So any \( x \) in \( \mathbb{R}^2 \) can be written as

\[
x = c_1 b_1 + c_2 b_2,
\]

so 

\[
Px = c_1 (1)b_1 + c_2 (-1)b_2 = c_1 b_1 - c_2 b_2.
\]

This formula shows that the mapping \( x \mapsto Px \) is a reflection across the line (eigenspace) \( L = \text{Span}\{b_1\} \) (see the figure below).