Review: $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ linear

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \ldots \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \quad \text{(in } \mathbb{R}^n)$$

Then find $T(e_1)$, $T(e_2)$, $T(e_n)$ (in $\mathbb{R}^m$)

Form matrix $A$:

$$A = \begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix}$$

called the standard matrix for $T$. $A$ must be $m \times n$

$$T(\mathbf{x}) = A\mathbf{x} = A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1T(e_1) + \ldots + x_nT(e_n)$$

$T(\mathbf{x})$ can always be written in this form as a matrix vector product.
**Example** (details discussed in class)

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the mapping that first projects a point $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ into the $yz$ plane (by changing its first coordinate to 0) and then rotates around the origin in the $yz$ plane by an angle of $\frac{\pi}{4}$. (*Draw a picture to illustrate.*)

<table>
<thead>
<tr>
<th>Vector $e_1$</th>
<th>Then projection onto $y$ plane $yz$ plane</th>
<th>Then rotated by $\frac{\pi}{4}$ around origin in $yz$ plane</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\begin{bmatrix} 1 \ 0 \ 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \ 0 \ 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \ 0 \ 0 \end{bmatrix} = T(e_1)$</td>
</tr>
<tr>
<td>$\begin{bmatrix} 0 \ 1 \ 0 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \ \frac{\sqrt{2}}{2} \ \frac{\sqrt{2}}{2} \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \ \frac{\sqrt{2}}{2} \ \frac{\sqrt{2}}{2} \end{bmatrix} = T(e_2)$</td>
</tr>
<tr>
<td>$\begin{bmatrix} 0 \ 0 \ 1 \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \ -\frac{\sqrt{2}}{2} \ \frac{\sqrt{2}}{2} \end{bmatrix}$</td>
<td>$\begin{bmatrix} 0 \ -\frac{\sqrt{2}}{2} \ \frac{\sqrt{2}}{2} \end{bmatrix} = T(e_3)$</td>
</tr>
</tbody>
</table>

The standard matrix $A = \begin{bmatrix} T(e_1) & T(e_2) & T(e_3) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

so $T(\mathbf{x}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\sqrt{2}}{2}y - \frac{\sqrt{2}}{2}z \\ \frac{\sqrt{2}}{2}y + \frac{\sqrt{2}}{2}z \end{bmatrix}$
A transformation $T$ is called **onto** if for every possible $b \in \mathbb{R}^m$, the equation $T(x) = b$ (or $Ax = b$) has at least one solution.

An onto transformation is also called a **surjection** (a term we won't use but you might come across it in doing WebWorK problems).
Suppose \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) \underline{linear}, with standard matrix \( A \), so that \( T(x) = Ax \)

The following are \underline{equivalent} (all true or all false)

\( T \) is called \underline{onto} if for \underline{every} possible \( b \in \mathbb{R}^m \)
there is at least one \( x \in \mathbb{R}^n \) for which \( T(x) = b \)

\( \Leftrightarrow \)

For every possible \( b \in \mathbb{R}^m \)
there is at least one \( x \in \mathbb{R}^n \) for which \( Ax = b \)

\( \Leftrightarrow \) \( \text{(see Theorem 4, p. 37)} \)

for \underline{every} \( b \in \mathbb{R}^m \), the equation \( T(x) = b \) has \underline{at least one} solution

\( \Leftrightarrow \)

for every \( b \in \mathbb{R}^m \), the equation \( Ax = b \) has at least one solution

\( \Leftrightarrow \) \( \text{(see Theorem 4, p. 37)} \)

every \( b \in \mathbb{R}^m \) is a linear combination of the columns of \( A \)

\( \Leftrightarrow \) \( \text{(see Theorem 4, p. 37)} \)

the columns of \( A \) span \( \mathbb{R}^m \)

\( \Leftrightarrow \) \( \text{(see Theorem 4, p. 37)} \)

\underline{Cannot be true} \quad \text{if } n < m \quad \rightarrow \quad A \text{ has a pivot position in every row}

For example, a linear transformation \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) \underline{cannot be onto} because

\[
\begin{bmatrix}
* & * \\
* & * \\
* & *
\end{bmatrix}
\]

\( \text{there cannot be a pivot in every row!} \)

An one-to-one transformation is also called an \underline{injection} (a term we won't use but you might come across it in doing WebWorK problems).
Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ linear, with standard matrix $A$ so that $T(\mathbf{x}) = A\mathbf{x}$

The following are **equivalent** (all true or all false)

\[ T \text{ is called one-to-one } \iff \text{ there is at most one } \mathbf{x} \in \mathbb{R}^n \text{ for which } T(\mathbf{x}) = \mathbf{b} \]

\[ \text{For every } \mathbf{b} \in \mathbb{R}^m, \text{ the equation } T(\mathbf{x}) = \mathbf{b} \text{ has at most one solution} \]

\[ \text{The equation } T(\mathbf{x}) = \mathbf{0} \text{ has only the trivial solution } \mathbf{x} = \mathbf{0} \]

\[ \Downarrow \quad \text{(See Theorem 11, p. 76)} \quad \Downarrow \]

\[ \Downarrow \quad \text{The homogeneous system } A\mathbf{x} = \mathbf{0} \text{ has only the trivial solution } \mathbf{x} = \mathbf{0} \]

\[ \Downarrow \quad \text{The system } A\mathbf{x} = \mathbf{0} \text{ has no free variables} \]

\[ \Downarrow \quad \text{(that is, every column of } A \text{ is a pivot column)} \]

\[ \Downarrow \quad \text{The columns of } A \text{ are linearly independent} \]

For example, a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ cannot be one-to-one because

the standard matrix is $2 \times 3 = \begin{bmatrix} \star & \star & \star \\ \star & \star & \star \end{bmatrix}$: there cannot be a pivot in every column!
Combing the last two red results, we see that for a linear transformation \( T : \mathbb{R}^n \to \mathbb{R}^m \) to be both one-to-one and onto, it is necessary to have \( n = m \) (so that the standard matrix is square.

In that case, there can be a pivot in every row and also a pivot in every column.

(In fact: when \( A \) is a square matrix, a pivot in every row is true if and only if there is a pivot in every column. So \( T : \mathbb{R}^n \to \mathbb{R}^n \) is automatically onto if it is one-to-one, and automatically one-to-one if it is onto. In this case, the rref of \( A \) must be \[
\begin{bmatrix}
? \\
\end{bmatrix}
\]
**Example** Suppose $T : \mathbb{R}^5 \to \mathbb{R}^5$ is linear, given by

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & -14 & 3 & 1 & 5 \\ -1 & 2 & 0 & 1 & 4 \\ 3 & 1 & 1 & 0 & 1 \\ -2 & -2 & 8 & 8 & 1 \\ -2 & 4 & 3 & 1 & 1 \end{bmatrix} \mathbf{x}$$

Row reducing $A$:

$$A = \begin{bmatrix} 2 & -14 & 3 & 1 & 5 \\ -1 & 2 & 0 & 1 & 4 \\ 3 & 1 & 1 & 0 & 1 \\ -2 & -2 & 8 & 8 & 1 \\ -2 & 4 & 3 & 1 & 1 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

There is a pivot in every row of $A$ so the linear transformation $T$ is onto
We discussed an example of a linear difference equation. Notes on that example were handed out in class and are also posted here. You should read those.