True or False (Handout Sheet)

1. Suppose that $A u = b$, $A v = b$ and $A z = 0$
   _____ a) $A (u + z) = b$
   _____ b) $A (u - v) = 0$
   _____ c) $A (u - v) = 0$ and therefore $u - v = z$

2. Suppose we have found two solutions $x$ and $y$ for the same equation: $Ax = b$ and $Ay = b$
   _____ We can get another solution to the same equation by just adding: $x + y$ must also be a solution.

3. Suppose $Ax = b$ and $Ay = 0$. Which of the following is true:
   _____ a) By adding, we can get another solution $x + y$ of the first equation
     b) By adding, we can get another solution $x + y$ of the second equation
Review

\( \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \} \) is linearly independent if

\[
x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + \ldots + x_p \mathbf{v}_p = \mathbf{0}
\]

has only the trivial solution

\[
\downarrow \uparrow
\]

\( \mathbf{x} = \mathbf{0} \)

has only the trivial solution

(where \( A = [ \mathbf{v}_1 \ \mathbf{v}_2 \ldots \mathbf{v}_p ] \))

\[
\downarrow \uparrow
\]

every column in \( A \) is a pivot column

(no free variables in equation \( A \mathbf{x} = \mathbf{0} \))

\[
\downarrow \uparrow
\]

\( A \) has linearly independent columns

\[
\downarrow \uparrow
\]

whenever \( c_1 \mathbf{v}_1 + \ldots + c_p \mathbf{v}_p = \mathbf{0} \)

all the weights (coefficients)

\( c_1, \ldots, c_p \) must be 0.
Logically the same thing, but stated for linearly dependent

\{v_1, v_2, ..., v_p\} is **linearly dependent** if

\[ x_1v_1 + x_2v_2 + \ldots + x_pv_p = 0 \]

has nontrivial solutions

\[ \downarrow \uparrow \]

\[ Ax = 0 \]

has nontrivial solutions

(where \( A = [v_1 \ v_2 \ \ldots \ v_p] \))

\[ \downarrow \uparrow \]

not every column in \( A \) is a pivot column

(so \( Ax = 0 \) has at least one free variable)

\[ \downarrow \uparrow \]

\( A \) has linearly dependent columns

\[ \downarrow \uparrow \]

it is possible to find weights \( c_1, \ldots, c_p \) not all 0

so that \( c_1v_1 + \ldots + c_pv_p = 0 \)
Important Results about linear dependence and independence
(for vectors in any $\mathbb{R}^n$)

(1,2,3 in blue are review)

1. $\{0\}$ is linearly dependent, but
   $\{\mathbf{v}_1\}$ is linearly independent if $\mathbf{v}_1 \neq 0$

2. Suppose $p \geq 2$. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent
   if and only if at least one of the vectors can be written as a linear combination of the others.

3. $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent
   if and only if one of the vectors is a scalar multiple of the other.

For the new items 4,5,6: see text or class notes explaining why each is true.

4. If any of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is $\mathbf{0}$, then
   $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent

5. If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are vectors in $\mathbb{R}^n$ and $p > n$ (“more vectors in the set than there are entries in each vector” — that is, the matrix $[\mathbf{v}_1 \ \mathbf{v}_2 \ldots \mathbf{v}_p]$ is “wider than it is tall”) then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ must be linearly dependent

For example: any 8 vectors in $\mathbb{R}^7$ are linearly dependent; and there cannot be 4 linearly independent vectors in $\mathbb{R}^3$.

6. (“The Preceding Vectors Theorem”)

   If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then (at least) one of the vectors must be a linear combination of the preceding vectors in the list.

(So: if $\mathbf{v}_1 \neq \mathbf{0}$ and no vector is a linear combination of the preceding vectors in the list, the vectors are linearly independent)
Is set $S$ linearly dependent or independent? Try to give more than one reason to justify your answer.

1. $S = \{ e_1, e_2, e_3, e_4 \}$

\[
\begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
1 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
0 \\
1 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix}
\]

2. $S = \left\{ \begin{bmatrix} \frac{7}{3} \\ \frac{4}{3} \\ 0 \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} -\frac{2}{3} \\ \frac{5}{13} \\ 0 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{6}{13} \\ \frac{6}{7} \\ -\frac{7}{13} \end{bmatrix} \right\}$

3. $S = \left\{ \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}$

4. $S = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

5. $S = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

Note: the $S$ is the set of pivot columns taken from the echelon form matrix matrix

\[
\begin{bmatrix}
2 & 4 & 3 & 0 & 1 \\
0 & 0 & 2 & -2 & 1 \\
0 & 0 & 0 & 5 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Explain why the set of pivot columns from an echelon matrix such as must always be a linearly independent set.