**Definition**  \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is called a **linear** transformation if

for all \( u, v \) in \( \mathbb{R}^n \), and all scalars \( c \)

1)  \[ T(u + v) = T(u) + T(v) \]  and

2)  \[ T(cu) = cT(u) \]

These same calculations work for any number of terms:

for example

\[ T(cu + dv - ew) = T(cu + dv) - T(ew) = cT(u) + dT(v) - cT(w) \]
Example
\[ T : \mathbb{R}^3 \to \mathbb{R}^2 \text{ where } T(x) = T( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} ) = \begin{bmatrix} |x_1| \\ x_2 \end{bmatrix}. \] Is \( T \) linear?

\[ u \quad v \quad u + v \]
\[ T(u + v) = T( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} ) = T( \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} ) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ but} \]

\[ T(u) + T(v) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}. \]

Since \( T(u + v) \neq T(u) + T(v) \), \( T \) is not linear; part 1) in the definition fails to be true.

Note that also \( T(-2u) = T( \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} ) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \neq \begin{bmatrix} -2 \\ 0 \end{bmatrix} = -2T(u) \), so that condition 2) in the definition of linear also fails to be true.

But to show that \( T \) is not linear, it is only necessary to show that one of the conditions 1), 2) in the definition

Can you find an example of a (nonlinear) mapping \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) where condition 1) is true but 2) is false? or where 2) is true but 1) is false?

Example \( T : \mathbb{R}^3 \to \mathbb{R}^3 \), \( T(x) = T( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} ) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \). Is \( T \) linear?

Geometrically \( T \) takes a point \( x \) in \( \mathbb{R}^3 \) and (by changing the third entry in the vector to 0) projects that point “straight down” to a point \( T(x) \) in the \( x_1-x_2 \) plane

Check algebraically whether 1) and 2) are both true.

1) If \( u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \) and \( v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \), then
\[ T(u) = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} \text{ and } T(v) = \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix}, \text{ so} \]

\[ T(u) + T(v) = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ 0 \end{bmatrix} \]

Now compute and compare:

\[ T(u + v) = T(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}) = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ 0 \end{bmatrix} \]

Also

2) \[ T(cu) = T(\begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix}) = \begin{bmatrix} cu_1 \\ cu_2 \\ 0 \end{bmatrix} = c \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} = cT(u) \]

Since 1) and 2) are both true, \( T \) is linear.

**Example**  Let \( A \) be an \( m \times n \) matrix and define a transformation

\[ T(\mathbf{x}) = A\mathbf{x} \quad (T \text{ is called a matrix transformation}) \]

\( T \) is linear because of properties we already know about the matrix-vector product:

\[ T(u + v) = A(u + v) = Au + Av \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \text{ and} \]

\[ T(cu) = A(cu) = cAu = cT(u) \]
For any linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$

Let $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, ..., $e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

For every $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, we can write $x = x_1 e_1 + x_2 e_2 + \ldots + x_n e_n$

Therefore

$$T(x) = x_1 T(e_1) + x_2 T(e_2) + \ldots + x_n T(e_n) \quad (*)$$

where (of course) each $T(e_i)$ is a vector in $\mathbb{R}^n$.

$(*)$ tells us a lot:

1) that every value $T(x)$ can be computed in we only know the values $T(e_1), T(e_2), \ldots, T(e_n)$

These $n$ (vector) values determine the value of $T(x)$ for all other values of $x$.

2) that we can always write a linear transformation $T$ as a matrix product:

$$T(x) = x_1 T(e_1) + x_2 T(e_2) + \ldots + x_n T(e_n)$$

$$= [T(e_1) \ T(e_2) \ldots \ T(e_n)] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A x,$$ where $A$ is the $m \times n$ matrix

whose columns are the vectors $T(e_1), T(e_2), \ldots, T(e_n)$. $A$ is called the standard matrix for the linear transformation $T$.

3) We can write down the standard matrix just by using the vectors $T(e_1), T(e_2), \ldots, T(e_n)$ as the columns for $A$. 

**Example** (see class notes/text for more details)

In the earlier example where $T: \mathbb{R}^3 \to \mathbb{R}^3$ was projection onto the $x_1$-$x_2$ plane:

$$T(e_1) = e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad T(e_2) = e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad T(e_3) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The standard matrix for the transformation is $A = [T(e_1) \ T(e_2) \ T(e_3)]$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

As a check, notice that $T(x) = Ax = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, just as before.

**Example** $T: \mathbb{R}^2 \to \mathbb{R}^2$, where $T(x) = \text{“the reflection of} \ x \text{across the line} \ y = x\text{”}$

We argued in class (on geometric grounds) that the transformation is linear.

It’s easy to see geometrically that $T(e_1) = e_1$ and $T(e_2) = e_1$. Therefore the standard matrix for this transformation is

$$A = [T(e_1) \ T(e_2)] = [e_1 \ e_1] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$ 

A formula for $T$ therefore is $T(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$.

The formula shows that reflecting a point (vector) across $y = x$ in $\mathbb{R}^2$ is the same as “reversing the coordinates.”

*(Recall from precalculus or calculus; if a function $y = f(x)$ has an inverse function $g$, the graph of the inverse function is found by reflecting the graph of $f$ across the line $y = x$. If a point $(a, b)$ is on the graph pf $f$, then $(b, a)$ is on the graph of $g$)*
Example  Let $T : \mathbb{R}^2 \to \mathbb{R}^2$, where $T(x) = \text{the vector obtained by rotating the vector } x\text{ around the origin through an angle } \theta$.

We argued in class (geometrically) that this transformation $T$ is linear.

Therefore, there is a $2 \times 2$ matrix $A$ such that $T(x) = Ax$ and $A = [T(e_1) \ T(e_2)]$.

(Draw a picture, as we did in class) The vector $e_1$ is represented by a arrow $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ from the origin. When $e_1$ is rotated by an angle $\theta$, the tip of the arrow moves along the unit circle to a new position which now represents the vector $\begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$; and the vector $e_2$ is moved to the vector with coordinates $\begin{bmatrix} \cos \left( \frac{\pi}{2} + \theta \right) \\ \sin \left( \frac{\pi}{2} + \theta \right) \end{bmatrix}$.

Therefore the matrix $A = \begin{bmatrix} \cos \theta & \cos \left( \frac{\pi}{2} + \theta \right) \\ \sin \theta & \sin \left( \frac{\pi}{2} + \theta \right) \end{bmatrix}$. This is “correct” but the formula can be simplified using trig identities:

\[
\begin{align*}
\cos \left( \frac{\pi}{2} + \theta \right) &= \cos \left( \frac{\pi}{2} \right) \cos \theta - \sin \left( \frac{\pi}{2} \right) \sin \theta = -\sin \theta \\
\sin \left( \frac{\pi}{2} + \theta \right) &= \sin \left( \frac{\pi}{2} \right) \cos \theta + \cos \left( \frac{\pi}{2} \right) \sin \theta = \cos \theta
\end{align*}
\]

so $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$.

Of course, for a specific angle $\theta$, the entries in $A$ are specific numbers:

for example, $T(x) = \begin{bmatrix} \cos \frac{\pi}{4} & -\sin \frac{\pi}{4} \\ \sin \frac{\pi}{4} & \cos \frac{\pi}{4} \end{bmatrix} x = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ represents a counterclockwise ( = positive angle ) rotation around the origin through an angle $\frac{\pi}{2} \ (90^\circ)$

Notice that this rotation moves $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and moves $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

A clockwise rotation by $\frac{\pi}{2}$ requires using $\theta = -\frac{\pi}{2}$. The matrix for this linear transformation would be

\[
A = \begin{bmatrix} \cos \left( -\frac{\pi}{2} \right) & -\sin \left( -\frac{\pi}{2} \right) \\ \sin \left( -\frac{\pi}{2} \right) & \cos \left( -\frac{\pi}{2} \right) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\]

What are the vectors $T(e_1)$ and $T(e_2)$?