For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ : what is the image of a line segment? of a parallelogram determined by two vectors?

Suppose $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{p} \in \mathbb{R}^n$ : The set of points $\mathbf{p} + t\mathbf{v}$ are a line in $\mathbb{R}^n$. The line contains the point $\mathbf{p}$ (let $t = 0$).

The set of points $\mathbf{p} + t\mathbf{v}$ where $t$ is restricted, say $0 \leq t \leq 1$, are a line segment in $\mathbb{R}^n$. The segment starts at $\mathbf{p}$ (when $t = 0$) and ends at $\mathbf{p} + \mathbf{v}$ (when $t = 1$).

Its image is the set of all points $T(\mathbf{p} + t\mathbf{v}) = T(\mathbf{p}) + tT(\mathbf{v})$ in $\mathbb{R}^m$.

These points form \[ \text{a line segment in from } T(\mathbf{p}) \text{ to } T(\mathbf{v}) \text{ in } \mathbb{R}^m \text{ if } T(\mathbf{v}) \neq \mathbf{0} \]

(this line segment goes from $T(\mathbf{p})$ to $T(\mathbf{v})$)

the point $T(\mathbf{p})$ if it happens that $T(\mathbf{v}) = \mathbf{0}$

Loosely speaking, if we think of a single point as a “degenerate” line or line segment, (length 0), then we could say:

for a linear transformation $T$: the image of a line segment is a line segment

the image of a line (no restriction on $t$) is a line.

The image of a parallelogram Suppose $\mathbf{u}, \mathbf{v}$ are linearly independent (so neither is a scalar multiple of the other). Then:

the set of all linear combinations $s\mathbf{u} + t\mathbf{v}$ \( (0 \leq s \leq 1 \text{ and } 0 \leq t \leq 1) \)

is the parallelogram determined by $\mathbf{0}, \mathbf{u}$ and $\mathbf{v}$.

Its image in $\mathbb{R}^m$ is the set of all points

$sT(\mathbf{u}) + tT(\mathbf{v})$ \( (0 \leq s \leq 1 \text{ and } 0 \leq t \leq 1) \)

These form a parallelogram in $\mathbb{R}^m$ (possibly “degenerate" with area 0).

\[
\begin{cases}
\text{parallelogram} & \text{if } T(\mathbf{u}), T(\mathbf{v}) \text{ are linearly independent, or} \\
\text{line segment} & \text{if } T(\mathbf{u}), T(\mathbf{v}) \text{ are linearly dependent and not both } \mathbf{0} \\
\text{point} & \text{if both } T(\mathbf{u}) \text{ and } T(\mathbf{v}) \text{ are } \mathbf{0}
\end{cases}
\]

See picture for an illustration with $T : \mathbb{R}^2 \to \mathbb{R}^2$. 
In the picture the red parallelogram (upper right) is determined by \(0, \mathbf{u}, \text{ and } \mathbf{v}\). A typical point in the parallelogram is \(s\mathbf{u} + t\mathbf{v}\) (the corner of the green parallelogram). Its image (drawn on the same coordinate grid) is the red parallelogram (lower left) determined by \(0, T(\mathbf{u}) \text{ and } T(\mathbf{v})\).

Every point inside the original parallelogram (see green parallelogram, \(s\mathbf{u} + t\mathbf{v}\)) has an image inside the other parallelogram (see green parallelogram at lower left, \(sT(\mathbf{u}) + tT(\mathbf{v})\)).

We can say that the image of a parallelogram under a linear transformation is a parallelogram provided we remember that the image might be a “degenerate” parallelogram (a line segment, or a point). This is true even if the parallelogram doesn't have a vertex at \(0\) = see below.
Linear transformation applied to a parallelogram with one vertex at \( p \)

Adding \( p \) to each point in a parallelogram simply “translates” the parallelogram by \( p \)

\[
\begin{align*}
\vec{p} + s\vec{u} + t\vec{v} \\
\text{translate by} \quad \vec{p} \quad \text{parallelogram vertex at 0}
\end{align*}
\]

The image is the set of all points

\[
\begin{align*}
T(p) + sT(u) + tT(v) \\
\text{translate by} \quad T(p) \quad \text{parallelogram containing 0}
\end{align*}
\]

See picture for an illustration with \( T : \mathbb{R}^2 \to \mathbb{R}^2 \). The image parallelogram is drawn in the same figure.
Examples $T : \mathbb{R}^2 \to \mathbb{R}^2$ linear,

i) $T(x) = \text{reflection across the } x_2\text{-axis}$. Here $T(e_1) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = -e_1$ and 
   
   $T(e_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e_2$. The standard matrix is $[T(e_1) \ T(e_2)]$.

ii) $T(x) = \text{reflection across the line } x_2 = x_1$. Here $T(e_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e_2$ and 
   
   $T(e_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e_1$. The standard matrix is $[T(e_1) \ T(e_2)]$.

See the pictures below. In each case, the parallelogram ( = “the unit square”) formed by $e_1$ and $e_2$ are transformed by $T$ into the blue parallelogram in the figure.
Example: Suppose $T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \mathbf{x}$, where $k > 0$. Show the effect of this transformation on the unit square.

We know that the unit square with two sides $\mathbf{e}_1$ and $\mathbf{e}_2$ is mapped to the parallelogram with sides $T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} k \\ 1 \end{bmatrix}$. (See the right-hand picture below). This transformation is called a horizontal shear to the right.

A formula is $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + kx_2 \\ x_2 \end{bmatrix}$

The case $k \leq 0$ is also illustrated below.
For \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \)

\[
\begin{array}{c}
\text{domain of } T \\
\text{codomain of } T
\end{array}
\]

the range is the set of all those \( b \)'s in \( \mathbb{R}^m \) that are the image of some \( x \) from \( \mathbb{R}^n \)

More precisely

\[
\text{range of } T = \{ b \text{ in } \mathbb{R}^m : b = T(x) \text{ for some } x \text{ in } \mathbb{R}^n \}
\]

The range is a subset of the codomain \( \mathbb{R}^m \) but might or might not equal \( \mathbb{R}^m \).

For example, if \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) is projection onto the \( x_1 \)-\( x_2 \) plane: 

\[
T(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}) = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}
\]

then (for example) \( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) is in the codomain \( \mathbb{R}^3 \) but not in the range of \( T \) (why?)

**Definition** A linear transformation \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is called **onto** if the range = all of \( \mathbb{R}^m \) (the codomain).
Suppose \( T : \mathbb{R}^n \to \mathbb{R}^m \) linear, with standard matrix \( A \), so that \( T(x) = Ax \).

**The following are equivalent (all true or all false)**

- \( T \) is called **onto** if for every \( b \in \mathbb{R}^m \) there is at least one \( x \in \mathbb{R}^n \) for which \( T(x) = b \).
- For every \( b \in \mathbb{R}^m \) there is at least one \( x \in \mathbb{R}^n \) for which \( Ax = b \).

\[ \iff \]

(see Theorem 4, p. 37)

for every \( b \in \mathbb{R}^m \), the equation \( T(x) = b \) has at least one solution

for every vector \( b \in \mathbb{R}^m \) is a linear combo of the columns of \( A \).

\[ \iff \]

(see Theorem 4, p. 37)

the columns of \( A \) span \( \mathbb{R}^m \).

\[ \iff \]

(see Theorem 4, p. 37)

\( A \) has a pivot position in every row.

\( \uparrow \)

This version allows us to actually figure out whether a specific \( T(x) = A \) is onto (by using row reduction).

**Examples:**

1) Suppose \( T : \mathbb{R}^5 \to \mathbb{R}^5 \), where \( T(x) = \begin{bmatrix} 2x_1 - 14x_2 + 3x_3 + x_4 + 5x_5 \\ -x_1 + 2x_2 + x_4 + 4x_5 \\ 3x_1 + x_2 + x_3 + x_5 \\ -2x_1 + 2x_2 + 8x_3 + 8x_4 + x_5 \\ -2x_1 + 4x_2 + 3x_3 + x_4 + x_5 \end{bmatrix} \). Is \( T \) onto?
Write $T(x) = Ax = \begin{bmatrix} 2 & -14 & 3 & 1 & 5 \\ -1 & 2 & 0 & 1 & 4 \\ 3 & 1 & 1 & 0 & 1 \\ -2 & -2 & 8 & 8 & 1 \\ -2 & 4 & 3 & 1 & 1 \end{bmatrix} x$

Row reducing $A = \begin{bmatrix} 2 & -14 & 3 & 1 & 5 \\ -1 & 2 & 0 & 1 & 4 \\ 3 & 1 & 1 & 0 & 1 \\ -2 & -2 & 8 & 8 & 1 \\ -2 & 4 & 3 & 1 & 1 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

There is a pivot in every row of $A$ so the linear transformation $T$ is onto

2) Can there be an onto linear transformation $T : \mathbb{R}^4 \rightarrow \mathbb{R}^6$?

If $T : \mathbb{R}^4 \rightarrow \mathbb{R}^6$ and $T(x) = Ax$, then $A$ is a $6 \times 4$ matrix. Since there are more rows than columns in $A$, there cannot be a pivot position in every row. Therefore a linear transformation $T : \mathbb{R}^4 \rightarrow \mathbb{R}^6$ cannot be onto.