Extra Problems on Centers of Mass (Centroids)  
Solution Sketches

1. Find the center of mass of the region between the graphs of \( f(x) = e^x \) and \( g(x) = -e^{2x} \), for \( 0 \leq x \leq 1 \).

The “top” boundary curve of the region is \( f(x) = e^x \) and the “bottom” boundary is \( g(x) = -e^{2x} \).

Therefore the mass of the region is \[ M = \int_0^1 \rho(f(x) - g(x)) \, dx = \int_0^1 \rho(e^x + e^{2x}) \, dx = \rho \left( e + e^2 - (1 + \frac{1}{2}) \right) = \rho \left( \frac{2e^2 + 3}{2} \right) \]

The moment around the \( y \)-axis is \[ M_y = \int_0^1 \rho x(e^x - (-e^{2x})) \, dx = \rho \int_0^1 xe^x + xe^{2x} \, dx \] Using integration by parts (once on each piece of the integrand) gives \[ \int xe^x + xe^{2x} \, dx = xe^x - e^x + \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} \]

The moment around the \( x \)-axis is \[ M_x = \int_0^1 \rho \left( \frac{1}{2}x^2 - \frac{1}{4}e^{2x} \right) \, dx = \frac{1}{2} \rho \int_0^1 e^{2x} - e^x \, dx = \frac{1}{2} \rho \left( \frac{1}{2}e^2 - \frac{1}{4}e^4 \right) \]

Therefore \( \bar{x} = \frac{M_y}{M} = \frac{\rho \left( \frac{2e^2 + 3}{2} \right)}{\rho \left( \frac{2e^2 + 3}{2} \right)} = \frac{e^2 + 5}{4} \cdot \frac{2}{2e + 2e^2 - 3} = \frac{e^2 + 5}{8e + 4e^2 - 6} \approx 0.63 \]

and \( \bar{y} = \frac{M_x}{M} = \frac{\rho \left( \frac{2e^2 + 4}{8} \right)}{\rho \left( \frac{2e^2 + 3}{2} \right)} = \frac{2e^2 - e^4 - 1}{8e + 4e^2 - 3} \cdot \frac{2}{2e + 2e^2 - 3} = \frac{2e^2 - e^4 - 1}{8e + 4e^2 - 12} \approx -1.04 \)

2. Without using symmetry arguments, show that the center of mass of a circle is at its center.

Suppose the circle is situated with center at \( (0, 0) \) and has equation \( x^2 + y^2 = r^2 \).

The top and bottom boundaries of the region are \( f(x) = \sqrt{r^2 - x^2} \) and \( g(x) = -\sqrt{r^2 - x^2} \).

The moment \( M_y \) is then \[ \rho \int_{-r}^r (f(x) - g(x)) \, dx = \rho \int_{-r}^r 2x \sqrt{r^2 - x^2} \, dx = \rho \left( \frac{2}{3} (r^2 - x^2)^{3/2} \right) \bigg|_{-r}^r = 0 \]

so \( \bar{x} = \frac{M_y}{M} = 0 \).
Also, 

\[ M_x = \frac{1}{2} \rho \int_{-\infty}^{0} f^2(x) - g^2(x) \, dx = \frac{\rho}{2} \int_{-\infty}^{0} (r^2 - x^2) - (r^2 - x^2) \, dx = \frac{\rho}{2} \int_{-\infty}^{0} 0 \, dx = 0, \]

so \( \bar{y} = 0. \)

Therefore the centroid is at \((0, 0) = \text{the center of the circle.}\)