Solutions to Differentiation Review Problems

1) (The derivatives are not been fully simplified)

a) \( \frac{dy}{dx} = \cos (\cos (\tan x)) \cdot (−\sin (\tan x)) \cdot \sec^2 x \)

b) \( \frac{dy}{dx} = \frac{1}{\sec^2 (x^2 + 1)} \cdot 2\sec(x^2 + 1) \cdot \sec(x^2 + 1) \cdot \tan(x^2 + 1) \cdot 2x \)
   \( = 4x \tan (x^2 + 1) \)
(or use properties of logarithms to simplify a bit before differentiating)

c) \( \frac{dy}{dx} = \ln 10 \cdot 10^{\arctan(2x)} \cdot \frac{1}{1+2x^2} \cdot 2x \cdot \ln 2 = \ln 10 \cdot 10^{\arctan(2x)} \cdot \frac{1}{1+2x^2} \cdot 2x \cdot \ln 2 \)

d) \( y = \log_7(xe^{\frac{\sqrt{2}}{x^2+1}}) = \log_7x + (x^2 + 1)^{1/3} \log_7e, \) so
\[
\frac{dy}{dx} = \frac{1}{x \ln 7} + \frac{1}{3} \log_7e \cdot (x^2 + 1)^{-2/3}(2x) = \frac{3(x^2+1)^{2/3}+2x^2}{(3x \ln 7)(x^2+1)^{2/3}}
\]

e) \( \ln y = \ln (x^{\arcsin(x^2)}) = \arcsin(x^2) \ln x, \) so
\[
\frac{dy}{dx} = y' = x^{\arcsin(x^2)} \left[ \frac{\arcsin(x^2)}{x} + (\ln x) \frac{2x}{\sqrt{1-x^4}} \right]
\]

f) \( \frac{dy}{dx} = \frac{(1+x^2) \cdot \frac{1}{1+x^2} - (\arctan x)(2x)}{(1+x^2)^2} = \frac{1-(\arctan x)(2x)}{(1+x^2)^2} \)

g) Since \( \frac{d}{dx} \ln |x| = \frac{1}{x}, \) the chain rule gives \( \frac{dy}{dx} = \frac{1}{\sin x} \cos x = \cot x \)

h) \( x^2y^2 + \sin y = x, \) so \( 2xy^2 + x^2(2y \frac{dy}{dx}) + (\cos y) \frac{dy}{dx} = 1, \) so \( \frac{dy}{dx} = \frac{1-2xy^2}{2x^2y + \cos y} \)

i) Since \( \ln(x^y) = \ln(y^x), \) we get \( y \ln x = x \ln y, \) so \( y \cdot \frac{1}{x} + (\ln x) \cdot y' = \ln y + x \cdot \frac{1}{y} \cdot y'. \)
Therefore \( y'(\ln x - \frac{1}{y}) = \ln y - \frac{y}{x}, \) which gives \( \frac{dy}{dx} = y' = \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}} = \frac{y(\ln y - y)}{x(y \ln x - x)} \)
2) a) The slope of the line segment $L$ is given by $\frac{f(b)-f(a)}{b-a}$.

b) In your graph, the tangent line to the graph at the point $(c, f(c))$ should be parallel to the line segment $L$.

c) The right hand side $\frac{f(b)-f(a)}{b-a}$ represents the average velocity of the car during the time interval $a \leq t \leq b$.

The left hand side represents the instantaneous velocity of the car at the moment $t = c$.

d) In this situation, the Mean Value Theorem says that at some time $c$ during your trip, your speedometer read exactly 100 km/hr. In other words, you can't average 100 km/hr for the trip without travelling at speed 100 km/hr at some time $t = c$ during the trip. (Maybe at many different times — but there must be at least one such time.)

3. a) Looking at the equation for the derivative $f'(x)$, we see that

\[
f'(x) > 0 \text{ if } x > 1 \quad \text{so } f(x) \text{ is increasing for } x > 1
\]
\[
f'(x) < 0 \text{ if } x < 1 \quad \text{so } f(x) \text{ is decreasing for } x < 1
\]

b) Since $f(x)$ decreases for $x < 1$ and increases for $x > 1$, it must have a local minimum at $x = 1$. There is no local maximum.

c) Differentiating we get that

\[
f''(x) = 2(x-1)(x-2) + (x-2)^2 = \]
\[= (x-2)(2(x-1) + (x-2)) = (x-2)(3x-4).
\]

Therefore

\[
f''(x) > 0 \text{ if } x > 2,
\]
so $f'(x)$ is increasing if $x > 2$,
so $f(x)$ is concave up if $x > 2$.

\[
f''(x) < 0 \text{ if } \frac{4}{3} < x < 2,
\]
so $f'(x)$ is decreasing if $\frac{4}{3} < x < 2$,
so $f(x)$ is concave down if $\frac{4}{3} < x < 2$.

\[
f''(x) > 0 \text{ if } x < \frac{4}{3},
\]
so $f'(x)$ is increasing if $x < \frac{4}{3}$,
so $f(x)$ is concave up if $x < \frac{4}{3}$.

Therefore $f(x)$ changes concavity at $x = \frac{4}{3}$ and $x = 2$. These correspond to inflection points on the graph.
4. a) 

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
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<tbody>
<tr>
<td>$s$</td>
<td>0</td>
<td>3</td>
<td>4</td>
<td>6</td>
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</tbody>
</table>

From the table, 

the average velocity over the time interval $[1, 2]$ is $\frac{f(2) - f(1)}{2 - 1} = 1 \text{ ft/sec}$

the average velocity over the time interval $[2, 3]$ is $\frac{f(3) - f(2)}{3 - 2} = 2 \text{ ft/sec}$.

These are two different estimates for the instantaneous velocity at time $t = 2$, one using data for times $t < 2$ and the other data for times $t > 2$.

Generally, we can get an improved estimate by averaging these two. So we estimate that the instantaneous velocity at time $t = 2$ is $\frac{2 + 1}{2} = 1.5 \text{ ft/sec}$.

b) We have estimated that $f'(2) \approx 1.5$. We might also write $\frac{ds}{dt} \bigg|_{t=2} \approx 1.5$, using the Leibniz notation for the derivative.