Part I, Multiple Choice, 5 points/problem: blacken your answers on the answer card.

1. A curve is given by \( \begin{cases} x = 1 + a \cos t \\ y = -2 + a \sin t \end{cases} \) \((0 \leq t \leq \frac{\pi}{2})\), where \(a\) is some positive constant. The length of the curve is 3. What is the value of \(a\)?

A) 1  B) \(\frac{1}{2}\)  C) \(\pi\)  D) \(\frac{\pi}{2}\)  E) 2

F) \(\frac{6}{\pi}\)  G) \(2\pi^2\)  H) \(2\pi\)  I) \(\frac{2\pi}{3}\)  J) \(3\)

Method I: We have \(x - 1 = a \cos t\) and \(y + 2 = a \sin t\). If we square both sides of each equation and add, we get \((x - 1)^2 + (y + 2)^2 = a^2 \cos^2 t + a^2 \sin^2 t = a^2\). So the point \((x, y)\) moves along the circumference of a circle or radius \(a\) centered at \((1, -2)\). For \(0 \leq t \leq \frac{\pi}{2}\), the point travels over a quarter of the circumference and that length \(= \frac{2\pi a}{4} = \frac{\pi a}{2}\). Therefore \(\frac{\pi a}{2} = 3\), which gives \(a = \frac{6}{\pi}\).

Method II: We have \(\frac{dx}{dt} = -a \sin t\) and \(\frac{dy}{dt} = a \cos t\). Since \(\frac{dx}{dt} \leq 0\) for \(0 \leq t \leq \frac{\pi}{2}\), the \(x\)-coordinate is always decreasing, so a point moving along a curve with those parametric equations moves along the path only once. Therefore “total distance traveled” = “length of curve” \(= \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \ dt = \int_0^{\pi/2} \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} \ dt = \int_0^{\pi/2} a \ dt = \frac{\pi a}{2}\). Therefore \(3 = \frac{\pi a}{2}\) or \(a = \frac{6}{\pi}\).
2. A (finite) region in the first quadrant is bounded by the line $y = x$ and the curve $y = x^3$. What is the volume of the solid obtained by revolving the region around the $x$-axis?

A) $\frac{3\pi}{4}$  B) $\frac{4\pi}{9}$  C) $\frac{7\pi}{24}$  D) $\frac{4\pi}{27}$  E) $\frac{3\pi}{8}$

F) $\frac{\pi}{7}$  G) $\frac{2\pi}{3}$  H) $\frac{11\pi}{3}$  I) $\frac{5\pi}{7}$  J) $\frac{5\pi}{9}$

The region is pictured below:

Using the “washer” method, the volume $V = \int_0^1 \pi (\text{outer radius}^2 - \text{inner radius}^2) \, dx$

$= \int_0^1 \pi (x^2 - (x^3)^2) \, dx = \pi \int_0^1 x^2 - x^6 \, dx = \pi \left( \frac{x^3}{3} - \frac{x^7}{7} \right) \bigg|_0^1 = \pi \left( \frac{1}{3} - \frac{1}{7} \right) = \frac{4\pi}{21}$
3. A spring is stretched 2 ft beyond its natural length and this requires 6 ft-lbs of work. How much additional work would be required to stretch the spring an additional foot?  

A) 3 ft-lbs  B) 3.5 ft-lbs  C) 4 ft-lbs  D) 4.5 ft-lbs  E) 5 ft-lbs  
F) 5.5 ft-lbs  G) 6 ft-lbs  H) 6.5 ft-lbs  I) 7 ft-lbs  J) 7.5 ft-lbs  

By Hooke's Law, the amount of force $F(x)$ needed for $x$ feet of "stretch" beyond the natural length of the spring is $F(x) = kx$. The work done in stretching 2 feet beyond the natural length is $W = \int_0^2 kx \, dx = \frac{k x^2}{2} \bigg|_0^2 = 2k$. Therefore $k = 3$.  

The amount of work done in stretching from 2 feet beyond natural length to 3 feet beyond natural length is 

$$W = \int_2^3 \frac{3}{2} x^2 \, dx = \frac{3}{2} \frac{x^3}{3} \bigg|_2^3 = 3 \left(\frac{9}{2} - \frac{4}{2}\right) = \frac{15}{2} = 7.5 \text{ ft-lbs}$$

4. What is the length of the graph of $y = \frac{4}{3} x^{3/2}$, where $0 \leq x \leq 2$?  

A) $\frac{13}{3}$  B) $\frac{12}{7}$  C) $\sqrt{2} - 1$  D) $\frac{\sqrt{2} - 1}{2}$  E) $\sqrt{2} - 1$  
F) $\frac{5}{7}$  G) $\frac{9}{4}$  H) $\frac{9}{2}$  I) $2\sqrt{2} + 1$  J) $2\sqrt{3} - 1$  

Method I: $L = \int_0^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_0^2 \sqrt{1 + (2x^{1/2})^2} \, dx = \int_0^2 \sqrt{1 + 4x} \, dx$.  

Letting $u = 1 + 4x$, $\frac{1}{4}du = dx$ gives $\int_0^2 \sqrt{1 + 4x} \, dx = \frac{1}{4} \int_0^9 \sqrt{u} \, du = \frac{1}{4} \cdot \frac{2}{3} u^{3/2} \bigg|_1^9 = \frac{20}{6} = \frac{13}{3}$.  

Method II: Let $x = t$, $y = \frac{4}{3} t^{3/2}$. Then $L = \int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt = \int_0^2 \sqrt{1 + 4t} \, dt$.  

Then the integral works out (just as above) = $\frac{13}{3}$.  

5. A tank 10 ft. tall has the shape of a circular cylinder; its radius is 5 ft. It contains water to a depth of 2 ft. How much work is done pumping the water up over the top edge of the tank? (Water weighs 62.5 lbs/ft$^3$; round your answer to the nearest integer. All answers are given in ft-lbs).  

A) 148,742  B) 101,257  C) 98,175  D) 95,332  E) 88,357  
F) 83,753  G) 79,841  H) 77,981  I) 75,369  J) 73,224  

Let the $y$-axis $(0 \leq y \leq 10)$ be the vertical axis of the cylindrical tank. At height $y$, the horizontal cross-section of the liquid (tank) is a circle, whose area $A(y) = \pi (5)^2 = 25\pi$ ft$^2$. For $0 \leq y \leq 2$, we think of a “thin slice” of the liquid at height $y$ (with thickness, say, $dy$). The slice has volume $A(y) \, dy$ ft$^3$, and the liquid in that slice weighs $62.5 A(y) \, dy = (62.5)(25\pi) \, dy$ lbs. That slice must be lifted approximately a distance $10 - y$ to get its liquid out of the tank. The total work is therefore $W = \int_0^2 (62.5)(25\pi)(10 - y) \, dy =$
6. The base of a certain solid is pictured. Cross-sections of the solid perpendicular to the $y$-axis and to the base are quarter-circles. What is the volume of the solid?

For $0 \leq y \leq 4$, the radius (pictured) of the circular cross-section is $x = \sqrt{y}$. Therefore the area of the cross-section at $y$ is

$$\frac{1}{4} \pi (\text{radius})^2 = \frac{1}{4} (\pi (\sqrt{y})^2) = \frac{\pi y}{4}.$$

Therefore the solid has volume

$$V = \int_0^4 A(y) \, dy = \frac{\pi}{4} \int_0^4 y \, dy = \frac{\pi}{4} \left| y^2 \right|_0^4 = \frac{\pi}{4} (8 - 0) = 2\pi.$$
7. A woman on an island teeming with hungry mosquitos is being bitten at random times. Assume that the random variable \( X \) = “time between bites” has an exponential density function. If the average time between bites is 1 minute, what is the probability that the time between bites is two minutes or more? (Round your answer to 4 decimal places.)

A) 0.0031  B) 0.0132  C) 0.1353  D) 0.1478  E) 0.1798  
F) 0.2234  G) 0.2437  H) 0.2539  I) 0.2754  J) 0.2971

The exponential density function is \( f(t) = \begin{cases} 0 & t < 0 \\ ce^{-ct} & t \geq 0 \end{cases} \) where \( c = \frac{1}{\mu} = \frac{1}{1} = 1 \)

Therefore \( P(X \geq 2) = \int_2^\infty f(t) \, dt = \int_2^\infty e^{-t} \, dt = \lim_{x \to \infty} -e^{-t}|_2^x = 0 - e^{-2} = e^{-2} \approx 0.1353 \)

8. For the same woman, what is the median time between bites? (All answers are given in minutes, and rounded to 4 decimal places.)

A) 0.0042  B) 0.1397  C) 0.1459  D) 0.3452  E) 0.5512  
F) 0.6931  G) 0.8734  H) 1.2234  I) 1.3457  J) 1.5243

The median time \( m \) is that time for which \( P(X \geq m) = 0.5 = P(X \leq m) \). Therefore

\[
0.5 = \int_{-\infty}^{m} f(t) \, dt = \int_{0}^{m} e^{-t} \, dt = -e^{-t}|_0^m = 1 - e^{-m}
\]

Therefore \( e^{-m} = 0.5 \), so \( m = -\ln(0.5) \approx 0.6931 \)

9. Suppose \( a, b, c, d \) all are positive constants. Find \( \lim_{n \to \infty} \frac{an + b\ln(n^3)}{cn - d\ln(n^2)} \) (if it exists).
Consider the function \( f(x) = \frac{a + b \ln(x^2)}{cx + d \ln(x^2)} \). Since \( x \to \infty \), we can use L'Hopital's Rule to write \( \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{a + b \ln(x^2)}{cx + d \ln(x^2)} = \frac{a}{c} \).

Since \( x \to \infty \), \( \lim_{x \to \infty} \frac{a + b \ln(n^2)}{cn - d \ln(n^2)} = \frac{a}{c} \) also.

10. For what \( x \)'s does the series
\[
-x + \frac{1}{4}x^3 - \frac{1}{16}x^5 + \frac{1}{64}x^7 - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{4^n}
\]
converge?

A) \(-1 < x < 1\)  B) \(-1 \leq x \leq 1\)  C) \(-2 < x < 2\)  D) \(-2 \leq x \leq 2\)

E) \(-4 < x < 4\)  F) \(-4 \leq x \leq 4\)  G) \(-\frac{1}{4} < x < \frac{1}{4}\)  H) \(-\frac{1}{4} \leq x \leq \frac{1}{4}\)

I) \(x = 0\) only  J) for no values of \( x \)

The series is a geometric series with ratio \( r = -\frac{x^2}{4} \). Therefore it converges when \(|r| = | -\frac{x^2}{4} | < 1\) and diverges when \(|r| = | -\frac{x^2}{4} | \geq 1\). This means the series converges exactly when \(-2 < x < 2\).

11. Suppose a random variable \( X \) has probability density function
\[
f(x) = \begin{cases} 0 & \text{for } x < 0 \text{ or } x > 2 \\ x & \text{for } 0 \leq x \leq 1 \\ 2 - x & \text{for } 1 < x < 2 \\ \end{cases}
\]
What is the mean \( \mu \) of \( X \)?

A) 0  B) 0.2  C) 0.4  D) 0.6  E) 0.8

F) 1  G) 1.2  H) 1.4  I) 1.6  J) 1.8

Method I: The graph of \( f(x) \) is symmetric around the vertical line \( x = 1 \). Therefore the region under the density function would “balance” on the line \( x = 1 \), so \( \mu = 1 \).

Method II: By definition, \( \mu = \int_{-\infty}^{\infty} xf(x) \, dx = \int_{0}^{2} x f(x) \, dx \)
\[
= \int_{0}^{1} x(x) \, dx + \int_{1}^{2} x(2 - x) \, dx = \frac{x^3}{3} \bigg|_{0}^{1} + (x^2 - \frac{x^3}{3}) \bigg|_{1}^{2} = \left( \frac{1}{3} - 0 \right) + \left( 4 - \frac{8}{3} - 1 + \frac{1}{3} \right)
\]
\[
= 1
\]
12. Suppose \( \sum_{n=1}^{\infty} (-1)^{n+1}a^n = a - a^2 + a^3 - \ldots = -\frac{1}{3} \). What is \( a \) ?

A) \( \frac{1}{2} \)  
B) 3  
C) \( \frac{1}{4} \)  
D) \( \frac{1}{5} \)  
E) \( \frac{1}{6} \)  
F) \( -\frac{1}{2} \)  
G) \( -\frac{1}{3} \)  
H) \( -\frac{1}{4} \)  
I) \( -\frac{1}{5} \)  
J) no such \( a \) exists

This is a geometric series with ratio \( r = -a \). Since it converges, we know \(|r| = |1 - a| < 1 \) and its sum is \( \frac{\text{first term}}{1 - \text{ratio}} = \frac{a}{1 + a} \).

Therefore \( \frac{a}{1 + a} = -\frac{1}{3} \). Solving this gives \( a = -\frac{1}{4} \).

13. The sequence defined recursively by \( \begin{cases} a_1 = 3 \\ a_{n+1} = 7 + \frac{1}{a_n} \end{cases} \) for \( n \geq 1 \) has a limit \( L > 0 \). What is \( L \) ?

A) 1  
B) 0  
C) \( \sqrt{7} \)  
D) 3 + \( \sqrt{7} \)  
E) \( \sqrt{137} - \sqrt{21} \)

F) \( \frac{5 + \sqrt{86}}{2} \)  
G) \( \frac{7 + \sqrt{53}}{2} \)  
H) \( \frac{3 + \sqrt{21}}{2} \)  
I) \( \frac{2 + \sqrt{21}}{7} \)  
J) \( \frac{5 + \sqrt{32}}{4} \)

Since \( \lim_{n \to \infty} a_n = L \), we know that \( \lim_{n \to \infty} a_{n+1} = L \) also (that is, if \( a_1, a_2, a_3, \ldots, a_n, \ldots \to L \) then \( a_2, a_3, a_4, \ldots, a_{n+1}, \ldots \to L \)).

Therefore \( L = \lim_{n \to \infty} a_{n+1} = 7 + \frac{1}{\lim_{n \to \infty} a_n} = 7 + \frac{1}{L} \). Since \( L = 7 + \frac{1}{L} \), we have \( L^2 - 7L - 1 = 0 \) and the quadratic formula gives \( L = \frac{7 \pm \sqrt{49 + 4}}{2} = \frac{7 \pm \sqrt{53}}{2} \). Since we are told that \( L \) is positive, it must be that \( L = \frac{7 + \sqrt{53}}{2} \).
14. One of the things you measured in Lab 4 was the change in temperature of a temperature probe that you had previously cooled in ice water. The model for the rate of change of temperature $T$ of the probe, based on Newton’s Law of Cooling, was the differential equation \[ \frac{dT}{dt} = -k(T - R). \] (Notice the “-” preceding the “$k$” !) $k$ and $R$ were assumed to be constants. In that situation, which of the following statements are true?

i) $T - R$ is positive
ii) the graph of $T$ has the line $y = R$ as a horizontal asymptote
iii) the graph of $T$ is increasing
iv) $k$ is negative

A) i), ii) only  B) i), iii) only  C) i), iv) only  D) ii), iii) only
E) ii), iv) only  F) iii), iv) only  G) i), iii), iii) only  H) i), ii), iv) only
I) ii), iii), iv) only  J) all are true

In the situation described, the constant $R$ was room temperature, and $R >$ temperature $T$ of cooled probe. (So i) is false). As time passes, temperature $T$ increases, eventually approaching room temperature $R$. (So ii), iii) are true).

Since the temperature is increasing, \( \frac{dT}{dt} \) must be positive. Since $T - R$ is negative, \((-k)\) must also be negative, that is, $k$ must be positive. (So iv) is false.)

Part II, True or False, 1 point each. Blacken your answers on the answer card.

15. Suppose $X$ is a normal random variable with mean $\mu$ and standard deviation $\sigma$. Then $P(X \leq \mu + \sigma) = P(X \geq \mu - \sigma)$.

A) True  B) False

The graph of the density function is symmetric around the vertical line at $x = \mu$, so the area under the graph to the left of $\mu + \sigma$ is the same as the area under the graph to the right of $\mu - \sigma$.

16. Suppose the sequence $\{a_n\}_{n=1}^{\infty}$ is increasing and each $a_n \leq 179$. Then, for some number $L$, \( \lim_{n \to \infty} a_n = L \).

A) True  B) False

If a sequence is increasing and bounded above (in this case, by 179), it must converge.
17. The integral test could be used to conclude that the series \( \sum_{n=1}^{\infty} \frac{n}{2} \) diverges.

A) True \hspace{1cm} B) False

The function \( f(x) = \frac{x}{2} \) is not decreasing, so the integral test does not apply. (Although, in fact, the series does diverge – for example, because \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n}{2} \neq 0 \).)

18. A population \( P \) is growing according to a logistic model. If the carrying capacity is 5000 individuals and the initial population is 50, then for times \( t \) near 0, the population grows almost exponentially.

A) True \hspace{1cm} B) False

Initial growth in a logistic model is nearly exponential if the initial population is small relative to the carrying capacity. (If \( K \) is the carrying capacity, then when \( t \) is near 0, the term \( (1 - \frac{P}{K}) \approx 1 \). Therefore \( \frac{dP}{dt} = kP(1 - \frac{P}{K}) \approx kP \), and \( \frac{dP}{dt} = kP \) is the differential equation describing exponential growth.

19. The equation \( \frac{dP}{dt} = kP(5 - \frac{P}{1000}) \) describes a logistic model (with the equation written in a slightly nonstandard form). In this model the carrying capacity is 1000.

A) True \hspace{1cm} B) False

As \( P \to \) carrying capacity, \( \frac{dP}{dt} \to 0 \) in a logistic model. Here, \( \frac{dP}{dt} \to 0 \) as \( P \to 5000 = \) carrying capacity.

Alternatively, you could see this by rewriting the equation in the textbook's “standard” form: \( \frac{dP}{dt} = 5kP(1 - \frac{P}{5000}) = KP(1 - \frac{P}{5000}) \).
Part III: These are two “free response” problems (#20, #21) worth a total of 25 points. Write your answers on the test pages. Show your work neatly and cross out irrelevant scratchwork, false starts, etc.

Please put your name on each of the following pages, since they may be separated during grading. Also, please add your Discussion Section Letter (available on your exam’s front cover sheet) so that we can return papers through discussion sections.

Name_______________________________   Discussion Section Letter _________

20. a) Does the series \( \sum_{n=1}^{\infty} \frac{n^2+1}{5n^2+5n+7} \) converge? Explain why or why not.

The series diverges because \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n^2+1}{5n^2+5n+7} = \lim_{n \to \infty} \frac{1+\frac{1}{n^2}}{5+\frac{5}{n}+\frac{7}{n^2}} = \frac{1}{5} \neq 0 \)

b) Write down the first 5 terms of a geometric series that does not converge.

Pick any \( a \neq 0 \) and any \( r \) with \( |r| \geq 1 \). Then \( a + ar + ar^2 + ar^3 + ar^4 + \ldots \) is correct.

For example: \( 1 + 2 + 4 + 8 + 16 + \ldots \) or \( \frac{3}{2} + \frac{12}{6} + \frac{48}{18} + \frac{192}{54} + \frac{768}{162} + \ldots \)

c) Explain why the Integral Test does or does not apply to the series \( \sum_{n=1}^{\infty} \frac{n}{e^n} \).

Then (using the integral test or some other test) decide whether the series converges or diverges.

The function \( f(x) = \frac{x}{e^x} = xe^{-x^2} \) is positive and continuous for \( x \geq 1 \). It is also decreasing for \( x \geq 1 \) because \( f'(x) = \frac{e^x-x(2xe^x)}{e^{2x}} = \frac{e^x(1-2x)}{e^{2x}} < 0 \) when \( 2x > 1 \), that is, when \( x > \frac{1}{\sqrt{2}} \approx 0.707 \). Therefore the Integral Test applies.

\( \int_{1}^{\infty} xe^{-x^2} \, dx \) converges – since \( \int_{1}^{\infty} xe^{-x^2} \, dx = \left[ -\frac{1}{2}e^{-x^2} \right]_{1}^{\infty} = 0 - \left( -\frac{1}{2}e^{-1} \right) = \frac{1}{2e} \).

By the Integral Test, the series also converges.
21. We are on the planet Erehwon, where the acceleration due to gravity is 20 m/sec$^2$. A mass of living tissue is being slowly lifted at a rate of 1 m/day. At the beginning, the tissue has mass 1 kg, but it is growing at a rate of 0.5 kg/day.

   a) Write the integral that represents the amount of work done if the mass is lifted 10 m. (Ignore the weight of the very lightweight super-strong rope that's being used.)

Consider a vertical $y$-axis, with $y = 0$ ground level. We are to lift the mass from $y = 0$ to $y = 10$. The force required at height $y$ to keep lifting is nothing but the “weight” of the mass (since we're ignoring the rope's weight).

At height $y$ (m), $y$ days have passed (at a rate of 1 m/day) and by then the mass has increased by $0.5y$ to a total mass of $1 + 0.5y$ (kg). Since $F = ma$, the weight of the mass (force due to gravity) is $(1 + 0.5y)(20)$ (kg-meters/sec$^2$ = newtons).

Therefore the work $W = \int_0^{10} (1 + 0.5y)(20) \, dy$

   b) Evaluate the integral. (Include units on the answer.)

\[
W = \int_0^{10} (1 + 0.5y)(20) \, dy = \int_0^{10} 20 + 10y \, dy \\
= 20y + 5y^2 \bigg|_0^{10} = 200 + 500 - 0 - 0 = 700 \text{ (joules)}
\]

c) Would the integral have been different if, on Erehwon, the acceleration due to gravity had been only 10 m/sec$^2$? If so, how would the integral in a) and the answer in b) have changed?

Yes: the “20” in the integral would instead have been “10” (m/sec$^2$) (i.e., the mass would have weighed only half as much), and the answer in b) would have been 350 joules.