Proofs by Mathematical Induction

"Mathematical induction" can be a useful way to prove that some statement (equation, inequality) is true for every value of \( n = 1, 2, 3, 4, \ldots \). In our study of sequences, it's sometimes helpful in proving that a sequence is monotonic (increasing or decreasing) or in proving that a sequence is bounded. But before we look at sequences, let's consider the general idea of mathematical induction.

Suppose we want to prove that the statement

\[
1 + 2 + \ldots + n = \frac{n(n+1)}{2} \quad (**) 
\]

is true for every value of \( n = 1, 2, 3, \ldots \). In other words, there's an infinite list of things we want to prove, one for each value of \( n \):

\[
n = 1 : \quad 1 = \frac{1(1+1)}{2} \quad \text{(call this equation } P_1) \\
n = 2 : \quad 1 + 2 = \frac{2(2+1)}{2} \quad \text{(call this equation } P_2) \\
n = 3 : \quad 1 + 2 + 3 = \frac{3(3+1)}{2} \quad \text{(call this equation } P_3) 
\]

and so on for \( n = 4, 5, 6, \ldots \).

So we want to prove that every statement in the list \( P_1, P_2, P_3, \ldots, P_n, P_{n+1}, P_{n+2}, \ldots \) is true, that is, that the equation (**) is true for every value of \( n = 1, 2, 3, \ldots \).

Here's the idea of a proof by induction:

a) First, we prove that if any statement in the list is true, then the next one in the list must also be true.

b) Second, we prove (often just by simple arithmetic) the first statement \( P_1 \) is true.

If we do both these things, what follows? We've checked that \( P_1 \) is true. But, by a), that means \( P_2 \) must also be true. But if \( P_2 \) is true then, by a), \( P_3 \) must also be true. But if \( P_3 \) is true, then, by a), \( P_4 \) must also be true...and so on. In other words, all the \( P_n \)'s must be true.

(Analogy: Imagine the statements \( P_n \) as being an infinite sequence of dominoes lined up one behind the other. Then a) says "we check that if any domino falls, then the next one must also fall," and b) says "we knock over the first domino." The conclusion is that "they all fall down."
Example  Prove (***) is true for all \( n = 1, 2, 3, \ldots \) using mathematical induction.

a) We pick the \( n^{th} \) statement in the list (\( n \) could be 1, 2, 3, ...). It is

\[
P_n : \quad 1 + 2 + \ldots + n = \frac{n(n+1)}{2}.
\]

We need to prove that if \( P_n \) is true, then the next statement must also be true. The “next” statement is \( P_{n+1} \), the “same” equation but with \( n + 1 \) replacing \( n \).

\[
P_{n+1} : \quad 1 + 2 + \ldots + n + (n + 1) = \frac{(n+1)(n+2)}{2}
\]

To do so, we assume that \( 1 + 2 + \ldots + n = \frac{n(n+1)}{2} \) is true. Then we can then say

\[
1 + 2 + \ldots + n + (n + 1) = \frac{n(n+1)}{2} + (n + 1) = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}
\]

is true, i.e.,

\[
1 + 2 + \ldots + n + (n + 1) = \frac{(n+1)(n+2)}{2}, \text{ and that's just } P_{n+1}.
\]

b) We prove that \( P_1 \) is true: but that's trivial. For \( n = 1 \), \( P_1 \) just says \( 1 = \frac{1(1+1)}{2} \), which is true — just by simple arithmetic.

Therefore by mathematical induction, \( P_n \) is true for all \( n = 1, 2, 3, \ldots \) (To reiterate the general idea, \( P_1 \) is true; and a) says that if a statement is true, then the next one is true — so \( P_2 \) is true; so \( P_3 \) is true; so \( P_4 \) is true; and so on all the way down the line.)

\( \text{(Side comment: as discussed in class in Chapter 5, this particular equality can be proved in a very simple without using mathematical induction. If we write} \)

\[
S = 1 + 2 + 3 + \ldots + (n - 1) + n, \quad \text{then (in reverse order)}
\]

\[
S = n + (n - 1) + (n - 2) + \ldots + 2 + 1
\]

\( \text{Adding the two equations gives} \)

\[
2S = (n + 1) + (n + 1) + (n + 1) + \ldots + (n + 1) + (n + 1) = n(n + 1), \text{ so}
\]

\[
S = \frac{n(n+1)}{2}.
\]

\( \text{Induction is often an easier and more straightforward way to prove such an equation, but it isn't always needed!} \)

\textbf{Practice Exercises using Induction}

1) Prove that \( 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \) for all \( n = 1, 2, \ldots \)
2) Prove that \( \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^n} = 1 - \frac{1}{2^n} \) for all \( n = 1, 2, \ldots \)
3) Prove that \( 1 + n < e^n \) for all \( n = 1, 2, \ldots \)
Example involving limits of sequences

Suppose a sequence is defined recursively by

\[ a_1 = 1 \]
\[ a_2 = \sqrt{1 + 1} = \sqrt{2} \]
\[ a_3 = \sqrt{1 + \sqrt{2}} \]

\[ a_n = \sqrt{1 + a_{n-1}} \quad \text{(next term} = \sqrt{1 + \text{previous term})} \]

Does \( \lim_{n \to \infty} a_n \) exist (that is, does the sequence converge)? If it does, what is its limit?

By the Monotonic Sequence Theorem (p. 570), we know that \( \{a_n\} \) converges if it is both monotonic (either increasing or decreasing) and bounded. Graphing the sequence suggests that \( \{a_n\} \) is increasing and that all the terms \( a_n \) are \( < 2 \). We prove each of these "guesses" by mathematical induction.
i) Prove that the sequence is increasing: we want to prove \( a_1 < a_2, \ a_2 < a_3, \ a_3 < a_4, \ldots, a_n < a_{n+1}, \ldots \) etc., that is, we want to prove
\[
a_n < a_{n+1} \text{ for all } n = 1, 2, \ldots
\]

a) First we prove that if one of these statements is true \((a_n < a_{n+1})\), then the next one \((a_{n+1} < a_{n+2})\) must also be true.

Suppose \(a_n < a_{n+1}\). Then
\[
a_{n+2} = \sqrt{1 + a_{n+1}} \quad (\text{definition of } a_{n+2})
\]
\[
> \sqrt{1 + a_n} \quad (\text{since } a_n < a_{n+1})
\]
\[
= a_{n+1} \quad (\text{definition of } a_{n+1})
\]
so \(a_{n+1} < a_{n+2}\) is true.

We then prove that the first statement is true: \(a_1 < a_2\). But that just says \(1 < \sqrt{2}\) which is true since \(\sqrt{2} \approx 1.414\)

By mathematical induction, we conclude that \(a_n < a_{n+1}\) for all \(n = 1, 2, \ldots\)

ii) Prove that \(a_1 < 2, \ a_2 < 2, \ldots, a_n < 2, \ldots\) In other words, prove that
\[
a_n < 2 \text{ for all } n = 1, 2, \ldots
\]

First we prove that if one of these inequalities is true, then the next one is also true:

Suppose \(a_n < 2\). Then
\[
a_{n+1} = \sqrt{1 + a_n} \quad (\text{definition of } a_{n+1})
\]
\[
< \sqrt{1 + 2} \quad (\text{since } a_n < 2)
\]
\[
= \sqrt{3} \approx 1.732 < 2
\]
so \(a_{n+1} < 2\)

Then we prove that the first inequality is true: \(a_1 < 2\) — but that just says \(1 < 2\).

By mathematical induction we have proved that the sequence \(\{a_n\}\) is both increasing and bounded and therefore \(\{a_n\}\) converges, that is, \(\lim_{n \to \infty} a_n = L\) for some number \(L\).
Now that we know \( \{a_n\} \) has a limit \( L \), we can try to find it.

Since \( \lim_{n \to \infty} a_n = L \), then \( \lim_{n \to \infty} a_{n+1} = L \) also.

[If that's clear to you, skip this paragraph and go on. Otherwise read this. What is the sequence \( \{a_{n+1}\} \)? Listing some terms for \( n = 1, 2, 3, \ldots \) we see that it's just \( a_2, a_3, a_4, \ldots \)

Since \( a_1, a_2, a_3, a_4, \ldots \rightarrow L \), then clearly \( a_2, a_3, a_4, \ldots \rightarrow L \) too (whether \( a_1 \) is present or not doesn't effect where the sequence ultimately “goes to” as \( n \to \infty \)). So \( \lim_{n \to \infty} a_{n+1} = L \).

Informally, the point is that the sequences \( \{a_{n+1}\} \) and \( \{a_n\} \) are identical except for “where they start” and therefore have the same limit. The difference in appearance is just due to a “shift” in the subscript from “\( n \)” to \( n + 1 \)”

So \( L = \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \sqrt{1 + a_n} = \sqrt{1 + \lim_{n \to \infty} a_n} = \sqrt{1 + L} \).

So \( L = \sqrt{1 + L} \). Squaring both sides gives \( L^2 = 1 + L \), so

\( L^2 - L - 1 = 0 \). By the quadratic formula, \( L = \frac{1 \pm \sqrt{1 + 4}}{2} \), so

\( L = \frac{1 + \sqrt{5}}{2} \) or \( L = \frac{1 - \sqrt{5}}{2} \). Since all the \( a_n \)'s are \( > 0 \) and increasing, \( \lim_{n \to \infty} a_n \) can't be negative, so \( \lim_{n \to \infty} a_n = \frac{1 + \sqrt{5}}{2} \).