Part I, Multiple Choice, 5 points/problem: Circle the correct answer. Write your work on the test booklet. If you miss the question I will look at the work on your test booklet. If there's evidence of some progress toward solving the problem (not just a miscellany of jotted down formulas), you might receive some partial credit.

1. A sensor measures the temperature $T$ of a cup of cooling coffee every 5 minutes. After 30 minutes, the data collected is:

<table>
<thead>
<tr>
<th>$t$ (min)</th>
<th>$T$ ($^\circ$F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>200</td>
</tr>
<tr>
<td>5</td>
<td>175</td>
</tr>
<tr>
<td>10</td>
<td>155</td>
</tr>
<tr>
<td>15</td>
<td>140</td>
</tr>
<tr>
<td>20</td>
<td>130</td>
</tr>
<tr>
<td>25</td>
<td>125</td>
</tr>
<tr>
<td>30</td>
<td>122</td>
</tr>
</tbody>
</table>

Use the midpoint approximation $M_3$ to estimate the average temperature of the coffee during this time period. (Round your answer to the nearest integer.)

A) 141°  B) 142°  C) 143°  D) 144°  E) 145°  F) 146°  G) 147°  H) 148°  I) 149°  J) 150°

The average of $T(t)$ is given by $\frac{1}{30-0} \int_0^{30} T(t) \, dt$. To estimate the integral, we divide the interval $[0, 30]$ into 3 subintervals of equal length, with $\Delta x = 10$. The midpoints of the subintervals are 5, 15, 25, so $M_3 = (T(5) + T(15) + T(25)) \cdot \Delta x = 10(175 + 140 + 125) = 4400$, so the average value of $T(t)$ over $[0, 30] = \frac{1}{30} (4400) = 146.666... \approx 147^\circ$. 
2. Suppose we approximate \( \int_1^\infty \frac{1}{x^2} \, dx \) with the integral \( \int_1^a \frac{1}{x^2} \, dx \). What is the smallest value of \( a \) for which the \(|\text{ERROR}| = |\int_1^\infty \frac{1}{x^2} \, dx - \int_1^a \frac{1}{x^2} \, dx| \leq 0.001\)?

A) 6  
B) \(5\sqrt{3}\)  
C) \(5\sqrt{10}\)  
D) 10  
E) \(\sqrt{20}\)  
F) \(\sqrt{30}\)  
G) \(10\sqrt{5}\)  
H) 20  
I) \(20\sqrt{2}\)  
J) \(4\sqrt{10}\)

\[
\int_1^\infty \frac{1}{x^2} \, dx \text{ converges, and for any } a > 1, \int_1^\infty \frac{1}{x^2} \, dx = \int_1^a \frac{1}{x^2} \, dx + \int_a^\infty \frac{1}{x^2} \, dx \text{ so } \left|\int_1^\infty \frac{1}{x^2} \, dx - \int_1^a \frac{1}{x^2} \, dx\right| = \int_a^\infty \frac{1}{x^2} \, dx = \lim_{t\to\infty} \frac{1}{t^a} = \frac{1}{a^2}, \\
\text{Therefore we need } \frac{1}{a^2} \leq 0.001 = \frac{1}{1000}, \text{ so } 2a^2 \geq 1000 \text{ or } a \geq \sqrt{500} = 10\sqrt{5}.
\]

3. Suppose a point's motion along a path is described by a set of parametric equations

\[
\begin{align*}
x &= \sin(at) \\
y &= \cos(at)
\end{align*}
\]

for \(0 \leq t \leq \pi\).

During this time interval, the point traveled 8 units along the path. What is the value of \(a\)?

A) \(\pi\)  
B) \(2\pi\)  
C) \(\frac{\pi}{2}\)  
D) \(\frac{2}{\pi}\)  
E) \(\frac{2}{2\pi}\)  
F) \(\frac{\sqrt{2}\pi}{2}\)  
G) \(\frac{8}{\pi}\)  
H) \(\frac{\pi}{10}\)  
I) \(\frac{3}{4\pi}\)  
J) \(\frac{3}{2\pi}\)

The distance traveled along the path is

\[
s = \int_0^\pi \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \, dt = \int_0^\pi \sqrt{(\cos(at))^2 + (-a\sin(at))^2} \, dt = \int_0^\pi \sqrt{a^2(\cos^2(at) + \sin^2(at))} \, dt = \int_0^\pi a \, dt = a\pi = 8, \text{ so } a = \frac{8}{\pi}.
\]
4. If we use the method of partial fractions to break up \( \frac{x+1}{(x^2+x+1)(x-3)} = \frac{Ax+B}{x^2+x+1} + \frac{C}{x-3} \), what is the value of \( C \)?

A) \( \frac{1}{2} \)  B) \( \frac{12}{7} \)  C) \( -\frac{2}{3} \)  D) \( \frac{4}{13} \)  E) \( \frac{2}{15} \)

F) \( \frac{1}{3} \)  G) \( -\frac{2}{7} \)  H) \( \frac{3}{8} \)  I) \( -\frac{7}{12} \)  J) \( -\frac{5}{3} \)

We have \( \frac{x+1}{(x^2+x+1)(x-3)} = \frac{Ax+B}{x^2+x+1} + \frac{C}{x-3} = \frac{(Ax+B)(x-3)+C(x^2+x+1)}{(x^2+x+1)(x-3)} \). Since the denominators of the left and right fractions are the same, the numerators must be equal:

\[ x + 1 = (Ax + B)(x-3) + C(x^2 + x + 1) \text{ for all } x. \]

Substituting \( x = 3 \), this becomes \( 4 = C(13) \), so \( C = \frac{4}{13} \). (Similarly, we could find \( A = -\frac{4}{13} \) and \( B = -\frac{3}{13} \)).

5. If you run the following m-file in Matlab, what is the result “ans” displayed in the workspace window?

\begin{verbatim}
 i=0;
 for n=1:5
   i=i+1;
   x=linspace(0,2,n);
   y=x.^2;
   dx=2/n;
   q(i)=sum(x.*y)*dx;
 end
 q(3)
\end{verbatim}

A) 0  B) 1  C) 2  D) 3  E) 4
F) 5  G) 6  H) 7  I) 8  J) 9

The variable “i” keeps track of the number of trips through the loop and the result of calculating the value of \( \sum(x.*y)*dx \) is stored in \( q(i) \) = the \( i^{th} \) entry in the array \( q \). \( q(3) \) is the valued computed for \( \sum(x.*y)*dx \) on the \( 3^{rd} \) trip through the loop, that is, when \( n=3 \).

When \( n=3 \), we have \( x=linspace(0,2,3)=[0,1,2] \), so \( y=[0,1,4] \). Then \( \sum(x.*y)=\sum([0,1,8])=9 \). Since \( dx=2/3 \), \( \sum(x.*y)*dx=9(2/3)=6 \).

6. Find the value of \( \int_{1}^{\infty} \frac{x+\sqrt{x}}{x^3} \, dx \) (if it converges).
\[ \int_1^\infty \frac{x + \sqrt{x}}{x^2} \, dx = \int_1^\infty \frac{x}{x^2} \, dx + \int_1^\infty \frac{\sqrt{x}}{x^2} \, dx = \int_1^\infty \frac{1}{x^2} \, dx + \int_1^\infty \frac{1}{x^{3/2}} \, dx = \lim_{t \to \infty} \left( -\frac{1}{x} \right)_1^t + \lim_{t \to \infty} \left( -\frac{2}{3} \frac{1}{x^{3/2}} \right)_1^t = 1 + \frac{2}{3} = \frac{5}{3}. \]
7. The base of a solid is the region in the plane above the $x$-axis and under the graph of $y = 4 - x^2$. The cross sections of the solid perpendicular to the $y$-axis are squares. What is the volume of the solid?

A) 16  B) 8  C) 24  D) 36  E) 45
F) 5  G) 12  H) 15  I) 14  J) 32

The base of the solid is pictured below. One edge of the cross-section of the solid, ⊥ to the $y$-axis at a point $y$ ($0 \leq y \leq 4$), is pictured. From the equation, its total width is $2\sqrt{4 - y}$. Therefore the area of the square cross-section at $y$ is $A(y) = (2\sqrt{4 - y})^2 = 4(4 - y)$. So the volume $V = \int_0^4 4(4 - y) \, dy = 4(y - \frac{y^2}{2})|_0^4 = 4((16 - 8) - (0 - 0)) = 32$ (units$^3$).
8. The graph pictured below is a piece of some cubic polynomial 
\[ y = f(x) = ax^3 + bx^2 + cx + d. \] We approximate \( \int_0^1 f(x) \, dx \) with 10 subdivisions using several different methods. Arrange the resulting approximations \( R_{10}, S_{10}, L_{10}, T_{10} \) in order of increasing size.

Since \( f^{(4)}(x) = 0 \), we know that 
\[ |\int_0^1 f(x) \, dx - S_{10}| \leq \frac{0(1-0)^5}{180(10)^5} = 0, \] that is, 
\[ S_{10} = \int_0^1 f(x) \, dx. \] Since \( f(x) \) is increasing, \( L_{10} < S_{10} < R_{10} \). Since the graph is concave up, \( T_{10} > \int_0^1 f(x) \, dx = S_{10} \). But since \( T_{10} \) is the average of \( L_{10} \) and \( R_{10} \), we know that \( T_{10} < R_{10} \).

Therefore \( L_{10} < S_{10} < T_{10} < R_{10} \).
9. A tank 20 m tall is shaped like a cylinder with a circular base. The base has radius 10 m. It is half full of water (which has a mass of 1000 kg/m³). How much work is done in pumping the water out of the tank by lifting it up to the top?

A) 2.32 × 10⁶ J  
B) 2.11 × 10⁶ J  
C) 4.11 × 10⁷ J  
D) 4.62 × 10⁶ J  
E) 1.34 × 10⁸ J  
F) 7.43 × 10⁷ J  
G) 5.13 × 10⁶ J  
H) 9.14 × 10⁷ J  
I) 7.22 × 10⁸ J  
J) 8.32 × 10⁸ J

Let \( y = 0 \) be the bottom of the tank and \( y = 20 \) the top. At any height \( y, \) \( 0 \leq y \leq 10, \) we look at a cross-sectional slice of the water. The slice is circular and has area \( A(y) = \pi \cdot 10² = 100\pi. \) Therefore a “thin slice” of the water has volume \( A(y) \, dy = 100\pi \, dy \) (m³) and the water in this slice has mass \( 1000 \cdot 100\pi \, dy = 10^5 \pi \) kg. The weight of this water is \( 9.8(10^5\pi \, dy) \) newtons, and it must be lifted approximately a distance of \( (20 - y) \) m to lift it to the top of the tank, which requires \( (20 - y) \cdot 9.8(10^5\pi \, dy) \) Joules of work. “Summing this up”, the work required is

\[
W = \int_0^{10} (20 - y) \cdot 9.8(10^5\pi \, dy) = 9.8(10^5\pi)\int_0^{10} 20 - y \, dy = 9.8(10^5\pi)(20y - \frac{y^2}{2})\bigg|_0^{10} = 9.8(10^5\pi)(150) \approx 4.62 \times 10^8 \text{ J}
\]

10. If \( a > 0, \) find \( \int_0^a \frac{\ln x}{x} \, dx \) (if the integral converges).

A) 0  
B) \frac{1}{a}  
C) 2a  
D) \frac{\ln a}{a}  
E) 2 \ln a  
F) a \ln a  
G) \frac{1}{2} \ln a  
H) \ln(2a)  
I) a \ln a - a  
J) integral diverges

The integral is improper because \( \frac{\ln x}{x} \) has a vertical asymptote at \( x = 0. \)

If we let \( u = \ln x, \) \( du = \frac{1}{x} \, dx, \) then

\[
\int \frac{\ln x}{x} \, dx = \int u \, du = \frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C.
\]

So

\[
\int_0^a \frac{\ln x}{x} \, dx = \lim_{t \to 0^+} \int_t^a \frac{\ln x}{x} \, dx = \lim_{t \to 0^+} \frac{1}{2}(\ln x)^2\bigg|_t^a = \lim_{t \to 0^+} \frac{1}{2}((\ln a)^2 - (\ln t)^2) = -\infty.
\]

So the integral diverges.
11. The semicircle pictured has center \( C \) at \( x = 5 \) on the \( x \)-axis. The region under the semicircle and above the line \( y = 3x \) is revolved around the \( x \)-axis. What is the volume of the resulting solid of revolution?

\[ \text{The circle with center } (5, 0) \text{ and radius } 5 \text{ has equation } (x - 5)^2 + y^2 = 25, \text{ so the semicircle pictured has equation } y = \sqrt{25 - (x - 5)^2}. \]

The semicircle and the line intersect where \( 3x = \sqrt{25 - (x - 5)^2} \). Squaring both sides we get \( 9x^2 = 25 - (x - 5)^2 = 25 - x^2 + 10x - 25 \), so that \( 10x^2 - 10x = 10x(x - 1) = 0 \) and \( x = 0 \) or \( x = 1 \).

The volume of the solid is given by

\[
\int_0^1 \pi ((\text{outer radius})^2 - (\text{inner radius})^2) \, dx = \int_0^1 \pi ((25 - (x - 5)^2 - 9x^2) \, dx = \int_0^1 \pi (10x - 10x^2) \, dx = 10\pi \left( \frac{x^2}{2} - \frac{x^3}{3} \right) \bigg|_0^1 = 10\pi \left( \frac{1}{2} - \frac{1}{3} \right) = \frac{10\pi}{6} = \frac{5\pi}{3} \text{ (units}^3\text{)}
\]
12. A piece of the text's Table of Integrals is attached as the last page of the exam booklet. (You can tear it off.) Use it to find the value of \( \int_{\ln 3}^{\ln 9} \frac{\sqrt{e^{2x} - 9}}{e^x} \, dx \).

A) \( \frac{9}{2} \ln 9 - 3 \sqrt{72} + \frac{9}{2} \ln 3 \)  

B) \( \frac{9}{5}(153) \sqrt{72} - \frac{9}{8} \ln (9 + \sqrt{72}) - \frac{9}{8} \ln 3 \)

C) \( \sqrt{72} - 3 \cos^{-1} (\frac{1}{3}) - \frac{3\pi}{4} \)  

D) \( \ln (9 + \sqrt{72}) - \frac{\sqrt{72}}{9} - \ln 3 \)

E) \( \ln(9 + \sqrt{72}) - \ln 3 \)

F) \( 3 \sqrt{72} + \frac{9}{2} \ln (9 + \sqrt{72}) - \frac{9}{2} \ln 3 \)

G) \( \frac{\sqrt{72}}{81} \)

H) \( \frac{3}{9 \sqrt{72}} - \frac{1}{\sqrt{72}} \)

I) \( \sqrt{72} \cos^{-1} 3 - \frac{3\pi}{4} \)

J) \( - \frac{1}{9} \ln \left( \frac{9 + \sqrt{72}}{9} \right) \)

To get a \( \sqrt{u^2 - 9} \) in the integral, we substitute \( u = e^x \), \( du = e^x \, dx \), so that \( dx = \frac{1}{u} \, du \). When \( x = \ln 9 \), \( u = e^{\ln 9} = 9 \), and when \( x = \ln 3 \), \( u = 3 \). Therefore

\[
\int_{\ln 3}^{\ln 9} \frac{\sqrt{e^{2x} - 9}}{e^x} \, dx = \int_3^9 \frac{\sqrt{u^2 - 9}}{u} \, du.
\]

Using Formula #42 from the table gives

\[
\int_3^9 \frac{\sqrt{u^2 - 9}}{u} \, du = - \frac{\sqrt{u^2 - 9}}{u} \left| u + \sqrt{u^2 - 9} \right| \bigg|_3^9 = - \frac{\sqrt{72}}{9} + \ln (9 + \sqrt{72}) - \ln 3.
\]

This is answer D) above. However note that the answer could have been simplified:

\[
- \frac{\sqrt{72}}{9} + \ln (9 + \sqrt{72}) - \ln 3 = - \frac{6 \sqrt{2}}{9} + \ln \left( \frac{9 + \sqrt{72}}{3} \right) = - \frac{2 \sqrt{2}}{3} + \ln \left( \frac{9 + 6 \sqrt{2}}{3} \right) = \ln (3 + 2 \sqrt{2}) - \frac{2 \sqrt{2}}{3}
\]
13. Find the length of the graph of \( y = x^2 - \frac{1}{8} \ln x \) over the interval \( 1 \leq x \leq 2 \).

(Hint: the expression that shows up under the square root is a perfect square.)

A) \( 2 - \ln 2 \)  
B) \( 3 + \ln 3 \)  
C) \( 1 + \ln 2 \)  
D) \( 3 + \frac{1}{8} \ln 2 \)  
E) \( \ln 2 \)  
F) \( 2 + \ln 3 \)  
G) \( 4 - \ln 3 \)  
H) \( 1 + \frac{3}{8} \ln 2 \)  
I) \( 2 + \frac{1}{4} \ln 2 \)  
J) \( 2 + \frac{1}{4} \ln 3 \)

\[
\frac{du}{dx} = 2x - \frac{1}{8x}, \text{ so } L = \int_1^2 \sqrt{1 + \left(\frac{du}{dx}\right)^2} \, dx = \int_1^2 \sqrt{1 + (2x - \frac{1}{8x})^2} \, dx = \\
\int_1^2 \sqrt{1 + 4x^2 - \frac{1}{2} + \frac{1}{64x^2}} \, dx = \int_1^2 \sqrt{4x^2 + \frac{1}{2} + \frac{1}{64x^2}} \, dx = \text{(use the hint, and the fact that this expression is so similar to the square } 4x^2 - \frac{1}{2} + \frac{1}{64x^2} \text{ in the preceding expression!)} \\
= \int_1^2 \sqrt{(2x + \frac{1}{8x})^2} \, dx = \int_1^2 2x + \frac{1}{8x} \, dx = (x^2 + \frac{1}{8} \ln x)|_1^2 = \\
(4 + \frac{1}{8} \ln 2) - (1 + \frac{1}{8} \ln 1) = 3 + \frac{1}{8} \ln 2.
\]

14. The integral \( \int_0^\infty \frac{1}{(4-2x)^{1/2}} \, dx \) is improper if \( 1 - 2k > 0 \), that is, if \( k < \frac{1}{2} \). For what values of \( k < \frac{1}{2} \) does the integral converge?

A) \( -1 < k < \frac{1}{2} \)  
B) \( -1 \leq k < \frac{1}{2} \)  
C) \( -2 < k < \frac{1}{2} \)  
D) \( -2 \leq k < \frac{1}{2} \)  
E) \( 0 < k < \frac{1}{2} \)  
F) \( 0 \leq k < \frac{1}{2} \)  
G) \( -\frac{3}{2} < k < \frac{1}{2} \)  
H) \( -\frac{3}{2} \leq k < \frac{1}{2} \)  
I) \( -\frac{1}{2} < k < \frac{1}{2} \)  
J) \( -\frac{1}{2} \leq k < \frac{1}{2} \)

Let \( u = 4 - 2x \), \( du = -2 \, dx \), \( dx = -\frac{1}{2} \, du \). Then \( \int_0^\infty \frac{1}{u^{1/2}} \, dx = \\
-\frac{1}{2} \int_4^0 \frac{1}{u^{1/2}} \, du = \frac{1}{2} \int_0^4 \frac{1}{u^{1/2}} \, du = \frac{1}{2} \int_0^4 u^{2k-1} \, du = \frac{1}{2} \lim_{t \to 0^+} \int_t^4 u^{2k-1} \, du.
\]

If \( k \neq 0 \) (that is, if \( 2k \neq 1 \neq -1 \)):

\[
\frac{1}{2} \lim_{t \to 0^+} \int_t^4 u^{2k-1} \, du = \frac{1}{2} \lim_{t \to 0^+} u^{2k-1} \bigg|_t^4 = 0.
\]

If \( 2k > 0 \), then \( t^{2k} \to 0 \) as \( t \to 0^+ \), so the limit exists: integral converges.

If \( 2k < 0 \), then \( t^{2k} \to \infty \) as \( t \to 0^+ \), so limit doesn't exist: integral converges.

If \( k = 0 : \frac{1}{2} \lim_{t \to 0^+} \int_t^4 u^{2k-1} \, du = \frac{1}{2} \lim_{t \to 0^+} \int_t^4 u^{1/2} \, du = \frac{1}{2} \lim_{t \to 0^+} u^{1/2} \bigg|_t^4 = \infty \), so the limit doesn't exist and the integral diverges.

Putting these observations together, the integral converges if \( 0 < k < \frac{1}{2} \).
Part II, True or False, 1 point each (no partial credit here)

15. Suppose the thin plate bounded \( y = f(x) \) and \( y = g(x) \) \((0 \leq x \leq \pi)\), pictured below, has center of gravity at \((\bar{x}, \bar{y})\). Then the region bounded by \( y = 2f(x) \) and \( y = 2g(x) \) also has center of gravity at \((\bar{x}, \bar{y})\).

A) True  
B) False

**False:** \( \bar{y} = \frac{\int_{0}^{\pi} \frac{1}{2}(f(x)^2 - g(x)^2) \, dx}{\int_{0}^{\pi} (f(x) - g(x)) \, dx} \) for the given region. If \( f \) and \( g \) are replaced by \( 2f \) and \( 2g \), then for the new “stretched” region, \( \bar{y}_{\text{new}} = \frac{\int_{0}^{\pi} \frac{1}{2}(2f(x)^2 - 2g(x))^2 \, dx}{\int_{0}^{\pi} (2f(x) - 2g(x)) \, dx} = \frac{4\int_{0}^{\pi} \frac{1}{2}(f(x)^2 - g(x)^2) \, dx}{2\int_{0}^{\pi} (f(x) - g(x)) \, dx} \)

\[= 2 \frac{\int_{0}^{\pi} \frac{1}{2}(f(x)^2 - g(x)^2) \, dx}{\int_{0}^{\pi} (f(x) - g(x)) \, dx} = 2\bar{y}.\]

16. The area between the graphs of \( y = \sin 4x \) and \( y = \tan x \) \((0 \leq x \leq \frac{\pi}{4})\) is given by \( \int_{0}^{\pi/4} \sin 4x - \tan x \, dx \).

A) True  
B) False

**False:** The graphs cross at some point \( b \approx 0.6 \). The area is given by \( \int_{0}^{b} \sin 4x - \tan x \, dx + \int_{\pi/4}^{b} \tan x - \sin 4x \, dx \).
17. The graph of \( y = f(x) \) is shown. If we approximate the integral \( \int_{0}^{6} f(x) \, dx \) with Simpson’s approximation \( S_6 \) and also with the trapezoidal approximation \( T_3 \), then \( T_3 = 3S_6 \).

**True:** \( S_6 = \frac{1}{3} (f(0) + 4f(1) + 2f(2) + 4f(3) + 2f(4) + 4f(5) + f(6)) \)

( the “\( \frac{1}{3} \)” is \( \frac{\Delta x}{3} \) )

But \( f(1) = f(3) = f(5) = 0 \), so in this case

\( S_6 = \frac{1}{3} (f(0) + 2f(2) + 2f(4) + f(6)) \)

\( T_3 = 1 \cdot (f(0) + 2f(2) + 2f(4) + f(6)) = (f(0) + 2f(2) + 2f(4) + f(6)) \)

( the “1” is \( \frac{\Delta x}{2} = \frac{2}{2} \) ),

so \( 3S_6 = T_3 \).
18. We can conclude that the integral \( \int_2^\infty \frac{e^{-x}}{3+x^2} \, dx \) converges by comparing it to the integral \( \int_2^\infty \frac{1}{x^p} \, dx \) which is known to converge.

\[ A) \text{True} \quad B) \text{False} \]

**True:** On \((2, \infty)\), \( e^{-x} \leq 1 \), so \( \frac{e^{-x}}{3+x^2} \leq \frac{1}{3+x^2} \leq \frac{1}{x^2} \). Since \( x \geq 2 \), we know \( x^5 \geq x^2 \) so \( \frac{1}{x^5} \leq \frac{1}{x^2} \). Since \( \int_1^\infty \frac{1}{x^p} \, dx \) converges whenever \( p > 1 \), we conclude that \( \int_2^\infty \frac{1}{x^p} \, dx \) converges. Therefore the integral \( \int_2^\infty \frac{e^{-x}}{3+x^2} \, dx \) converges since \( \int_2^\infty \frac{e^{-x}}{3+x^2} \, dx \leq \int_2^\infty \frac{1}{x^2} \, dx \).

19. The graph shows the velocity function \( v(t) \) for a point moving along a straight line. You are given that for \( 0 \leq t \leq 10 \), the displacement of the particle is 45 m. Then there are exactly six \( t \)-values at which the instantaneous velocity = average velocity for the trip.

\[ \text{Velocity of a point moving along a straight line} \]

\[ v \text{ (m/sec)} \]

\[ t \text{ (sec)} \]

\[ A) \text{True} \quad B) \text{False} \]

**True:** displacement = \( \int_0^{10} v(t) \, dt = 45 \), so average velocity = \( \frac{1}{10-0} \int_0^{10} v(t) \, dt = 45 = 4.5 \text{ m/sec} \), A horizontal line at height \( v = 4.5 \) intersects the graph of \( v(t) \) exactly 6 times.
Part III: These are “free response” problems worth a total of 25 points. Show your work neatly and cross out irrelevant scratchwork, false starts, etc.

20. Find \( \int \frac{1}{(x+1)(x+2)} \, dx \)

Use partial fractions: write

\[
\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} = \frac{A(x+2)+B(x+1)}{(x+1)(x+2)}
\]

Equating the numerators gives

\[
A(x + 2) + B(x + 1) = 1 \text{ for all } x.
\]

Substituting

\[
x = -2 \quad \text{gives } B(-1) = 1 \text{ so } B = -1
\]

\[
x = -1 \quad \text{gives } A(1) = 1 \text{ so } A = 1
\]

Then

\[
\int \frac{1}{(x+1)(x+2)} \, dx = \int \frac{1}{x+1} - \frac{1}{x+2} \, dx = \ln |x+1| - \ln |x+2| + C = \\
\ln |\frac{x+1}{x+2}| + C
\]
21. a) Without computing $S_2$ or evaluating the integral, explain why Simpson's approximation $S_2$ gives the exact value for $\int_7^{10} x^2 - 9x + 3 \, dx$.

**One reason:** $S_2$ estimates $\int_7^{10} f(x) \, dx$ by approximating $f(x)$ over $[7, 10]$ with a parabola. Since $f(x) = x^2 - 9x + 3$ is a parabola to begin with, the “fit” is perfect and $S_2 = \int_7^{10} x^2 - 9x + 3 \, dx$.

**Alternate reason:** For $f(x) = x^2 - 9x + 3$, $f^{(4)}(x) = 0$, so we can use $K = 0$ in Simpson's error control formula, getting $|\int_7^{10} x^2 - 9x + 3 \, dx - S_2| \leq 0$, i.e., $S_2 = \int_7^{10} x^2 - 9x + 3 \, dx$.

b) Give a specific example another function $y = f(x)$ which is neither linear nor quadratic but for which $S_2$ gives the exact value $\int_7^{10} f(x) \, dx$. Explain (without actually computing $S_2$ and $\int_7^{10} f(x) \, dx$) why this is true.

Let $f(x) = x^3 + x^2 + x + 1$ (or any cubic polynomial). As in the “alternate reason” in part a), $f^{(4)}(x) = 0$ and the error control formula again gives $|\int_7^{10} f(x) \, dx - S_2| \leq 0$, so $S_2 = \int_7^{10} f(x) \, dx$. 
c) Let \( f(x) = e^{-x/2} \). Find the fourth derivative \( f^{(4)}(x) \), and, on the grid below, draw a reasonable sketch of the graph of \(|f^{(4)}(x)|\) over the interval \([0, 1]\).

\[
\begin{align*}
 f'(x) &= -\frac{1}{2}e^{-x/2}, \\ f''(x) &= \frac{1}{4}e^{-x/2}, \\ f'''(x) &= -\frac{1}{8}e^{-x/2} \quad \text{and} \\ f^{(4)}(x) &= f^{(4)}(x) = \frac{1}{16}e^{-x/2}. 
\end{align*}
\]

Since \( f^{(4)}(x) \geq 0 \), \( f^{(4)}(x) = |f^{(4)}(x)| \). Its graph is shown below. The crucial thing is that it is decreasing and that for \( 0 \leq x \leq 1 \),
\[
|f^{(4)}(x)| \leq f^{(4)}(0) = \frac{1}{16} = 0.0625.
\]
d) If we approximate $\int_0^1 e^{-x/2} \, dx$ with Simpson's approximation $S_4$, then we can say that

$$|\text{ERROR}| = |\int_0^1 e^{-x/2} \, dx - S_4| \leq ?$$

(Find a value of ? that's as small as possible based on your graph above. Round your answer to 8 decimal places.)

Since $|f^{(4)}(x)| \leq \frac{1}{16} = K$, the Simpson “error control” formula gives

$$|\int_0^1 e^{-x/2} \, dx - S_4| \leq \frac{K(1-0)^5}{180(4^3)} \approx 0.00000136$$

e) Given that $S_4 = 0.78693975$ (rounded to 8 decimal places), use your result in d) to write an inequality stating:

$$?? \leq \int_0^1 e^{-x/2} \, dx \leq ??$$

(The values of ?? and ??? should be given rounded to 8 decimal places also.)

From b) (rounded to 8 places)

$$|\int_0^1 e^{-x/2} \, dx - S_4| \leq 0.00000136$$

$$-0.00000136 \leq \int_0^1 e^{-x/2} \, dx - S_4 \leq 0.00000136$$

$$S_4 - 0.00000136 \leq \int_0^1 e^{-x/2} \, dx \leq 0.00000136 + S_4.$$  Substituting $S_4 = 0.78693975$ gives

$$0.78693839 \leq \int_0^1 e^{-x/2} \, dx \leq 0.78694111$$