Background Information for Problem 1

In class, we did an example in which we solved a differential equation to find the velocity $v(t)$ when a body weighing 1 lb. was dropped and air resistance was taken into account. In that example, we assumed that the drag force (upward force due to air resistance) was proportional to the velocity: in fact, we assumed that the drag force $= \frac{1}{100} v$. We assumed the body was dropped from position $s = 0$ and that the positive axis was directed downward. After setting up and solving the differential equation and applying the condition $v(0) = 0$, we ended up with

$$v(t) = 100 - 100e^{-8t/25}$$

An interesting fact is that $\lim_{t \to \infty} v(t) = 100$ (ft/sec), that is, the body approaches a “limiting velocity” and is, in fact, very close to the limiting velocity by the time $t = 20$ sec (when $t = 20$, $v \approx 99.83$ ft/sec). (This assumes, of course, that the body was dropped from a position high enough that it hasn’t hit the ground before $t = 20$. It’s easy to find the formula for the position $s(t)$ from $v(t)$. It turns out that $s(20) \approx 1688$ ft, so that if the body were dropped from a height of, say, 2000 ft, it would still be falling when $t = 20$.)

In our first problem in this lab, we’ll apply a similar analysis to falling raindrops where, again, air resistance is an important part of analyzing the motion. (The raindrop material is adapted from The Calculus Reader, Smith & Moore, D.C. Heath and Co. 1992.)

Empirical evidence suggests that for very small falling spherical droplets (roughly, diameter $\lesssim 0.003$ inches), the drag force is proportional to the velocity $v$. In the discussion below, we do essentially the same analysis as we did for the falling body in class: only the constants are different.

Let’s assume such a drop weighs $w_0$ lbs and that it begins to fall at $s = 0$ with initial $v = 0$. We take “down” as the positive direction. Since its weight $= w_0 = F_{\text{gravity}} = ma = m(32)$, the mass of our drop is $m = \frac{w_0}{32}$.

The total force $F$ acting on the drop is

$$F = \text{downward force (lbs) due to gravity} + \text{upward force (drag) due to air resistance}$$

$$= w_0 - Cv$$

(where $C$ is some proportionality constant)

But the total force $F = ma = m \frac{dv}{dt} = \frac{w_0}{32} \frac{dv}{dt}$, so (equating the two expressions for $F$) we get

$$\frac{w_0}{32} \frac{dv}{dt} = w_0 - Cv, \text{ or } \frac{dv}{dt} = \frac{32}{w_0} (w_0 - Cv) = 32 - \frac{32C}{w_0} v = 32 - k v \text{ (where } k = \frac{32C}{w_0}).$$

In summary, the velocity must satisfy the differential equation

$$\frac{dv}{dt} = 32 - k v, \text{ where } k \text{ is some constant.}$$

We can now solve the differential equation by separating variables:

$$\frac{dv}{32 - kv} = dt$$
\[- \frac{1}{k} \ln |32 - kv| = t + A \quad \text{(where A is a constant of integration)}\]

\[
\ln |32 - kv| = -kt - kA \\
|32 - kv| = e^{-kt} - kA = e^{-kA}e^{-kt} \\
32 - kv = \pm e^{-kA}e^{-kt} = Ce^{-kt} \quad \text{(where constant } C = \pm e^{-kA})
\]

so

\[v = \frac{1}{k}(32 - Ce^{-kt})\]

Since \(v(0) = 0\)

\[0 = \frac{1}{k}((32 - C(1)), \text{ so } C = 32. \quad \text{Therefore} \]

\[v = \frac{32}{k}(1 - e^{-kt})\]

So (just like our example in class) for such a small falling drop there is a limiting velocity:

\[
\lim_{t \to \infty} v = \lim_{t \to \infty} \frac{32}{k}(1 - e^{-kt}) = \frac{32}{k}.
\]

In this situation, empirical measurements give that \(k \approx \frac{0.329 \times 10^{-5}}{D} \), where \(D\) is the diameter of the drop, measured in feet. If our drop has diameter 0.003 inches = 0.00025 feet, then the \(k\) value will be \(k \approx \frac{0.329 \times 10^{-3}}{(0.00025)^2} \approx 52.64\). Therefore the limiting velocity of the drop is \(\frac{32}{52.64} \approx 0.6079\) ft/sec \((\approx 0.4145\) mph) — a nice very gentle fall (as you actually observe with very small raindrops!)
Notice that the falling droplet reaches its limiting velocity very quickly (in fact $v \approx 0.6079$ ft/sec, rounded to 4 decimal places, by the time $t \approx 0.18$ sec)! For all practical purposes, the velocity of the raindrop is constant (0.6079 ft/sec) during its entire fall.

Falling constantly at velocity 0.6079 ft/sec from a height of 3000 ft., it would take the drop about $t = \frac{3000 \text{ ft}}{0.6079 \text{ ft/sec}} \approx 4935$ seconds $\approx 1.37$ hours.

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1. In this problem, we will investigate what happens with much larger raindrops. You’ll be using the program “euler”. Before you begin, go into file m-file “euler” and modify it so that the procedure only graphs a solution curve moving to the right of the initial point. (Deactivate the command “plot(xr,yr,’r’,xl,yl,’g’)” and activate the command “plot(xr,yr)”.

Use “short format” throughout this problem — more digits would be “phony” accuracy in such a modeling situation.

For large raindrops (diameter $D \geq 0.05$ inches), experimental evidence suggests that the differential equation discussed in the background information no longer describes the drop’s velocity very well. The data suggests, rather, that the drag force is proportional to the square of the velocity, which leads, in the same way as before, to a new differential equation describing the velocity:
\[ v' = 32 - cv^2 \]

where the proportionality constant \( c \approx \frac{0.000400}{D} \) (again, with distances in feet and time in seconds.)

a) Suppose a raindrop has diameter 0.05 inches and starts its fall with velocity 0. Use "euler" with \( dx = 0.01 \) to produce an approximate graph of the velocity function \( v \). You may need to experiment a little to decide how far to the right to extend the graph. Don't draw it a lot farther than you need to. Turn in the graph, including the limiting velocity (rounded to 4 decimal places) and the approximate time that velocity first appears to be reached. Also turn in the function m-file you used with Euler.

(In setting up the function m-file \( f \) for "euler", \( f \) is a function of \( t \) and \( v \).)

b) Repeat part a) with a raindrop with a diameter 4 times as large.

c) The fastest observed times for raindrops to fall 3000 feet are roughly between one and two minutes. Are your results in a) and b) consistent with those observations? Explain.

d) Finding the exact solution for the differential equation \( v' = 32 - cv^2 \) is messier than for the one in the introduction (although you should be able to do it using partial fractions). As a matter of fact, the solution (with \( v(0) = 0 \)) is

\[
v = \sqrt{\frac{32}{c}} \left( 1 - e^{-2\sqrt{32/D}t} \right)
\]

Given this solution, find the exact value of the limiting velocity in parts a) and b). (Compare with your approximate results using "euler" as a check.)

Note:  i) Dealing with raindrops of "intermediate size" seems to be a more complicated problem

ii) According to the guys on NPR's "Car Talk", the drag force on a moving car is also roughly proportional to the square of the velocity. The fact that drag increases with velocity helps explain why gas mileage drops off at higher speeds. Of course — up to a certain point — engine efficiency also increases with speed. Such considerations were why the speed limit on most interstates was lowered to 55 mph during the energy crisis of the late 70's.
Dear Tom and Ray:

Please help settle a dispute with a colleague. She insists that there is a negligible difference in the amount of gas used going 70 mph vs. 50 mph. I say there is an optimum gas saving speed, and for most cars, that is around 50 mph. All else being equal, and you have one gallon of gas in your car, and it's twenty-something miles through the desert to the next gas station, how fast would you drive? Thanks for the info. Jeff

Tom: Gee, Jeff. With all due respect, I'd have to say that your colleague has her headlight in her taillight socket. And I mean that in the most respectful way.

Ray: The biggest issue in gas mileage at higher speeds is wind resistance. And resistance increases by the square of the speed. If you do the math, you can see that 50 (mph) squared is 2500, and 70 (mph) squared is 4900. So the resistance at 70 mph is almost DOUBLE what it is at 50 mph. And that makes a big difference in fuel economy. That's why aerodynamics became so important in the last fifteen years, and why all of our cars started to look like jelly beans.

Tom: And to answer your other question, the greatest gas savings takes place when the engine is turning its slowest. The slower the engine turns, the fewer explosions in the cylinders. And the fewer explosions in the cylinders, the less gas that gets used to make those explosions.

Ray: So to get the greatest possible gas mileage, you want the engine to turn as slowly as possible, while the car moves as quickly as possible. And you accomplish that by running at low engine speed in the highest possible gear.

Tom: So if I were in the desert, and had one gallon of fuel to get me twenty miles, I'd accelerate very gently and slowly until the car shifted into its highest gear (assuming it's an automatic transmission). And the minute the engine speed dropped as that last shift took place, I'd stop accelerating and hold the speed right there. On most cars, that shift to overdrive would happen somewhere between 35 and 50 miles per hour. And--provided you stayed in high gear and didn't downshift--that would be the optimum speed for maximum fuel efficiency.

Ray: And if it were really important to save gas, you'd want to keep the windows closed, turn the accessories off, keep the pop-up headlights down...and snap off that wind-resistant hood ornament just for good measure.
2. In the early 1920's two American scientists, Raymond Pearl (a biologist and statistician) and Lowell Reed (a mathematician) worked on models of US population growth. One model they considered was the logistic model we discussed in class:

\[
\frac{dP}{dt} = kP \left(1 - \frac{P}{M}\right)
\]

for some constants \(k\) (representing the "initial relative growth rate") and \(M\) ("the carrying capacity"). If you like, you can look at their original paper *On the rate of growth of the United States Population since 1790 and its mathematical representation* in the Proceedings of the National Academy of Sciences (6)1920, 275-288.

Of course, Pearl and Reed had only the census data through 1920 to work with. We put ourselves in their position and try to model that data with a logistic curve. If you don't make any mistakes, you should come to the same conclusions they did in 1920.

Of course, we already know the solution of the differential equation:

\[
P = \frac{M}{1 + Ae^{-kt}}, \text{ where } A = \frac{M - P_0}{P_0} \quad (\ast)
\]

So our job is simply to estimate the constants \(M\), \(k\) and \(A\) from census data. Except for the final graphing, much of the work in this problem needs to be done by hand.

From the daily assignments page in the web syllabus, get a copy of the data file named "uspop". (Use Internet Explorer rather than Netscape: Netscape somehow corrupts the file when downloading it!) The data file “uspop” contains two variables named “t” and “pop” which contain census data for the US population. “t” contains times (measured in decades, from 1790) and “pop” contains the corresponding population (in millions).

a) (Part a) will take some time (by hand), so you may want to postpone it until you're out of the lab. You can proceed with b) by temporarily assuming the equation (** is true.)

To estimate the constant \(M\) we will only use the data prior to 1920 (as Pearl and Reed did). We choose two times spaced out across the time period: let's call them \(t_1 = 6\) and \(t_2 = 12\). Let \(P_1\) be the population at time \(t_1 (1850)\) and \(P_2\) the population at time \(t_2 (1910)\). As usual, \(P_0\) denotes the initial population at time \(0 (1790)\).

(Note: any two times for which we have the data could be used. However choosing two times spaced out across the period 1790-1920 have a better chance, one would think, of capturing what's going on with the population. I deliberately chose times so that \(t_2 = 2t_1\). That is not necessary, but it makes the algebra below easier.)

Following the hints below and giving all your steps, show that the following equation is true. Do not substitute the values for the letters along the way.

\[
(P_0P_2 - P_1^2)M^2 + (P_0P_1^2 + P_1^2P_2 - 2P_0P_1P_2)M = 0 \quad (**) \]

Hints: Substituting \( t_1 \) and \( t_2 \) into the solution (*) gives us

\[
P_1 = \frac{M}{1 + Ae^{-x_1}} \quad \text{and} \quad P_2 = \frac{M}{1 + Ae^{-x_2}}
\]

Solve each equation for \( t_1 \) and \( t_2 \) to get

\[
t_1 = ? \quad \text{and} \quad t_2 = ??
\]

Since you know that \( t_2 = 2t_1 \), you can then write

\[
?? = 2 (?)
\]

Then simplify the equation to the form requested above. This requires nothing but careful, patient algebra. (Notice that the \( k \)’s can be canceled out along the way.)

b) In the equation (**), you know the value of everything but \( M \). Solve the equation symbolically for \( M \). \( M = ???. \) (The equation is a quadratic equation in the variable \( M \) and easy to solve by factoring.)

i) Using a calculator or Matlab, substitute the values for \( P_0 \) (1790), \( P_1 \) (1850), \( P_2 \) (1910) to estimate the “carrying capacity” \( M \), as Pearl and Reed did in 1920. Turn in the estimate. (Check: if you’ve done everything right, \( M \) should come out “not too far” from 200 (million).)

ii) Calculate and turn in the estimated value for \( A \).

iii) Knowing \( M \) and \( A \), go back to your earlier equation \( t_1 = ? \) and substitute to find an estimated value for \( k \), the “initial relative growth rate.” Turn in the estimate. (Note: the other equation \( t_2 = ?? \) could also be used; it produces the same estimated \( k \) value.)

c) Using Matlab, plot the actual US population data, and also mark each data point with a cross ‘+’. Put a grid in the picture. Turn “hold on” so you can add another graph.

Using your values of \( M, k, A \), plot the solution (*) logistic model in the same figure. Add appropriate titles and label and turn in the picture.

d) You should see that “logistic growth” models the population quite well over the time period Pearl and Reed were studying. According to this model, what should have happened to the US population in the long run? The model seems to “break down” about when? Speculate about why.
See attached newspaper article from Klick and Klack