EXAMINATION 2 AND SOLUTIONS

MATH 1322

Do the following problems. The point value of each problem is indicated in parentheses. There are 30 points all together. Show your work or otherwise explain what you are doing. Write in complete sentences, especially when stating definitions and theorems. Begin each problem on a new page of your blue book. Put the problems in your blue book in the order given.

1. (5) Sketch the region enclosed by the curves \( x + y^2 = 2 \) and \( x + y = 0 \). Find the area of the region by setting up the appropriate integral and then evaluating it.

2. (4) Find the volume of the solid \( S \). The base of \( S \) is a circular disk with radius \( r \). Parallel planar cross-sections perpendicular to the base are equilateral triangles.

3. (4) Write a Matlab script which will approximate the length of the curve \( y = \sin x \), where \( 0 \leq x \leq \pi \), by calculating the length of an approximating polygonal path of 1,000 segments. (Notice the 1,000 segments require 1,001 points).

4. (3) Consider the upper half of an ellipse traced out by the parametric equations \( x = 3 \cos t, \ y = 2 \sin t \), for \( 0 \leq t \leq \pi \). Set up an appropriate integral for the area of the region bounded above by this curve and below by the \( x \)-axis. Then evaluate the integral.

5. Let \( f(x) = x/8 \) if \( 0 \leq x \leq 4 \), and \( f(x) = 0 \) otherwise.
   (a) (2) Explain why this function \( f \) is a probability density function.
   (b) (1) If \( X \) is a random variable whose probability density function is this function \( f \), find the probability that \( 2 \leq X \leq 3 \).
   (c) (2) Find the mean, \( \mu \), of \( f \).
   (d) (2) Find the average, \( f_{\text{ave}} \), of \( f \) over the interval \( 0 \leq x \leq 4 \).

6. (4) Find the \( y \)-coordinate, \( \bar{y} \), of the centroid of the triangle with vertices at \((0,0)\), \((1,0)\) and \((0,2)\). Set up an appropriate integral and evaluate it.

7. (3) A tank full of water is in the shape of a right circular cone, with base of radius 3 ft and vertex at the top at a height of 10 ft. Find the work needed to pump the water out through the vertex at the top. Set up an appropriate integral and evaluate it. The weight density of water is \( \delta = 62.5 \) lbs per \( \text{ft}^3 \).

Date: Monday, October 26, 1998.
1. Solutions

1. The curve \( x = 2 - y^2 \) is a parabola opening to the left, while \( y = -x \) is a line through the origin with slope \(-1\). They intersect where \( 2 - y^2 = x = -y \), so \( y^2 - y - 2 = 0 \), whose solutions are \( y = -1, 2 \). In the region the curve \( x = 2 - y^2 \) lies always to the right of the line \( x = -y \). Hence,

\[
\text{Area} = \int_{-1}^{2} (2 - y^2 - (-y)) \, dy = \left(2y - \frac{y^3}{3} + \frac{y^2}{2}\right) \bigg|_{-1}^{2} = 4.5
\]

2. Let the \( x \)-axis be a diagonal of the disk with the origin at the center. The vertical line through a point \( x \) is a chord of height \( y = \sqrt{r^2 - x^2} \). The planar cross section is an equilateral triangle of side length \( 2y = 2\sqrt{r^2 - x^2} \), and therefore of height \( \sqrt{4y^2 - y^2} = \sqrt{3}y \). The cross sectional area is \( A(x) = \frac{1}{2}2\sqrt{3}y = \sqrt{3}(r^2 - x^2) \). Therefore,

\[
\text{Volume} = \int_{-r}^{r} \sqrt{3}(r^2 - x^2) \, dx = \frac{4r^3}{\sqrt{3}}
\]

3. \texttt{x=linspace(0,pi,1001)};
\texttt{y=sin(x)};
\texttt{deltax=diff(x)};
\texttt{deltay=diff(y)};
\texttt{deltas=sqrt(deltax.^2+deltay.^2)};
\texttt{L=sum(deltas)}

4. For \( x = 3 \cos t, y = 2 \sin t, 0 \leq t \leq \pi \),

\[
\text{Area} = \int_{-\pi}^{\pi} y \, dx = \int_{0}^{\pi} \sin t \, d(3 \cos t) = 6 \int_{0}^{\pi} \sin^2 t \, dt = 3\pi = 9.4248
\]

5. (a) This function is a probability density function because it is defined and non negative on the whole real line and

\[
\int_{-\infty}^{\infty} f(x) \, dx = \int_{0}^{4} \frac{1}{8} x \, dx = 1
\]

(b) \( P(2 \leq X \leq 3) = \int_{2}^{3} f(x) \, dx = \int_{2}^{3} x/8 \, dx = 5/16 = .3125 \)

(c) The mean is

\[
\mu = \int_{-\infty}^{\infty} x f(x) \, dx = \int_{0}^{4} \frac{1}{8} x \, dx = \frac{8}{3} = 2.67
\]

(d) The average of \( f \) over the interval \([0, 4]\) is \( f_{\text{ave}} = \frac{1}{4} \int_{0}^{4} f(x) \, dx = 1/4. \)

6. \( \bar{y} = M_x/M, \) where \( M = \text{Area} = \frac{1}{2}12 = 1 \) and

\[
M_x = \int_{0}^{2} y(1 - \frac{1}{2}y) \, dy = \frac{2}{3}
\]

Therefore, \( \bar{y} = 2/3. \)
7. The horizontal planar cross section of the water at depth $x$ is a circular disk of radius $r$. By similar triangles, $r/x = 3/10$, and hence $r = .3x$. The work required to lift the infinitesimal horizontal slice of water at depth $x$ up to the vertex is

$$dW = \delta x \pi r^2 \, dx = \pi \delta .09x^2 \, dx$$

Therefore, the work needed to pump the water out through the vertex at the top is

$$W = \int_0^{10} dW = .09\pi \delta \int_0^{10} x^3 \, dx = 225\pi \delta$$

Putting $\delta = 62.5$, we have $W = 44,178.6$ ft-lbs.