equicontinuous=uniformly equicontinuous, for brevity in the following.

(1) If \(\{f_n\}\) is an equicontinuous sequence of functions on a compact interval and \(f_n \to f\) pointwise, prove that the convergence is uniform.

First, we prove that \(f\) is uniformly continuous on the interval as follows. Given \(\epsilon > 0\), there exists a \(\delta > 0\) such that \(|f_n(x) - f_n(y)| < \epsilon\) if \(|x - y| < \delta\). For such \(x, y\), fixed, we can find an \(N \gg 0\) so that \(|f(x) - f_N(x)| < \epsilon\) and \(|f(y) - f_N(y)| < \epsilon\), by pointwise convergence. So, we get,

\[
|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \leq 3\epsilon.
\]

Next we prove that the convergence is uniform. Given \(\epsilon > 0\), choose \(\delta > 0\) so that \(|f_n(x) - f_n(y)| < \epsilon\) and \(|f(x) - f(y)| < \epsilon\) if \(|x - y| < \delta\). Next, subdivide the interval into \(p\) intervals of length, say \(\delta/2\) and pick \(x_i\), \(1 \leq i \leq p\) in these intervals. Now, we can find an \(N \gg 0\) so that for all \(n \geq N\), \(|f_n(x_i) - f(x_i)| < \epsilon\) for all \(i\). If \(n \geq N\) and \(x\) is any point in the interval, choose an \(x_i\) so that both \(x, x_i\) belong to the same subdivision. Then,

\[
|f_n(x) - f(x)| \leq |f_n(x) - f_n(x_i)| + |f_n(x_i) - f(x_i)| + |f(x_i) - f(x)| \leq 3\epsilon.
\]

(2) If \(|f_n(x) - f_n(y)| \leq M|x - y|^{\alpha}\) for some fixed \(M\) and \(\alpha > 0\) and all \(x, y\) in a compact interval for all \(n\), show that \(\{f_n\}\) is equicontinuous.

Given \(\epsilon > 0\), choose \(\delta < (\epsilon/M)^{\frac{1}{\alpha}}\). (Note that if \(M = 0\), any choice of \(\delta\) will do.)

(3) Let \(\{f_n\}\) be a sequence of \(C^\infty\) functions on a compact interval such that for any integer \(k \geq 0\), there exists an \(M_k\) such that \(|f_n^{(k)}(x)| \leq M_k\) for all \(x\) and \(n\). Prove that there exists a subsequence converging uniformly, together with all its derivatives to a \(C^\infty\) function.

First, we prove that the sequence \(\{f_n^{(k)}\}\) is equicontinuous for all \(k\). We have \(f_n^{(k)}(x) - f_n^{(k)}(y) = f_n^{(k+1)}(z)(x - y)\) for some \(z\) between \(x, y\) by mean value theorem. Thus

\[
|f_n^{(k)}(x) - f_n^{(k)}(y)| = |f_n^{(k+1)}(z)||x - y| \leq M_{k+1}|x - y|,
\]

and the previous problem finishes the proof.

Now do the diagonal trick. By theorem proved in class, since \(\{f_n\}\) is equicontinuous and uniformly bounded (by \(M_0\)), we can find a subsequence \(f_{11}, f_{12}, \ldots\) which converges uniformly in the interval. The sequence \(\{f_{1n}'\}\) is also uniformly bounded and
equicontinuous, so we can extract a subsequence which is uniformly convergent. This means, there is a subsequence of \( \{ f_{1n} \} \), say \( \{ f_{2n} \} \) such that \( \{ f'_{2n} \} \) is uniformly convergent. We can continue this and get subsequences \( \{ f_{pn} \} \) such that \( \{ f'_{pn} \} \) is uniformly convergent. Finally, as in class, one checks that the subsequence \( f_{11}, f_{22}, \ldots \) has all the required properties.

(4) Prove that the set of all polynomials of degree at most \( N \) (fixed) and coefficients in \([-1, 1]\) is uniformly bounded and equicontinuous in any compact interval.

If \( P \) is any such polynomial and \( x \) is any point in our interval, we have \( P(x) = a_0 + a_1 x + \cdots + a_N x^N \) with \( |a_i| \leq 1 \) for all \( i \) and thus, \( |P(x)| \leq 1 + |x| + \cdots + |x|^N \) and the last term is a continuous function on a compact set and hence bounded, independent of the \( a_i, x \). So, we see that the family of these polynomials is uniformly bounded.

Again, if \( P \) is as above, for any two points \( x \neq y \) we have,

\[
|P(x) - P(y)| = |x - y||a_1 + a_2(x + y) + a_3(x^2 + xy + y^2) + \cdots + a_N \frac{x^N - y^N}{x - y}|
\]

\[
\leq |x - y|(1 + 2M + 3M^2 + \cdots + NM^{N-1})
\]

where \( M \) is a positive number such that \( |x| \leq M \) for all \( x \) in our compact interval. Rest is clear from an earlier problem.

(5) Prove that the family of polynomials \( P \) of degree at most \( N \) with \( |P(x)| \leq 1 \) on \([0, 1]\) is equicontinuous on \([0, 1]\).

We use a fact from the last homework. Let \( P_k \) be polynomials of degree at most \( N \) for \( 0 \leq k \leq N \) such that \( \int_0^1 P_k x^j dx = 0 \) for all \( j \neq k \) and \( 0 \leq j \leq N \) and \( \int_0^1 P_k x^k dx = 1 \). If \( P(x) = a_0 + a_1 x + \cdots + a_N x^N \) is any polynomial with \( |P(x)| \leq 1 \) for all \( x \in [0, 1] \), we see that \( a_k = \int_0^1 P_k P dx \) and thus \( |a_k| \leq \int_0^1 |P_k| dx = M_k \). So, all the coefficients are uniformly bounded and then the proof is exactly as in the previous problem.

(6) Let \( P_0 = 0 \) and \( P_{n+1}(x) = P_n(x) + \frac{x^2 - P_n^2(x)}{2} \).

(a) Prove that \( |x| - P_{n+1}(x) = (|x| - P_n(x)) \left(1 - \frac{|x| + P_n(x)}{2}\right)\). Deduce that \( 0 \leq P_n(x) \leq P_{n+1}(x) \leq |x| \) if \( |x| \leq 1 \). The formula is obvious.

\[
|x| - P_{n+1}(x) = |x| - P_n(x) - \frac{x^2 - P_n^2(x)}{2}
\]

\[
= (|x| - P_n(x)) \left(1 - \frac{|x| + P_n(x)}{2}\right)
\]
Assume we have proved the result for \( n \), the initial case being obvious. Then we need to prove that \( P_n(x) \leq P_{n+1}(x) \leq |x| \). Since \( 0 \leq P_n(x) \leq |x| \), we see that \( |x| \leq P_n(x) + |x| \leq 2|x| \) and thus,

\[
1 - \frac{|x|}{2} \geq 1 - \frac{|x| + P_n(x)}{2} \geq 1 - |x|.
\]

It is clear that this inequality implies what we need.

(b) Show that \( |x| - P_n(x) \leq |x|(1 - \frac{|x|}{2^n}) < \frac{2}{n+1} \) for \( |x| \leq 1 \).

The equation (1) above immediately implies the first part of inequality by induction. For the second part, check that the function \( x(1 - \frac{x}{2})^n \) attains its maximum at \( x = \frac{2}{n+1} \) in \([0, 1]\).

(c) Prove that \( \{P_n\} \) converges uniformly to the function \( g(x) = |x| \) in \([-1, 1]\).

Less said the better.