Answers to Homework 8

In the following, \((X, \mathcal{F}, \mu)\) will be a measure space.

(1) Let \(f_1 \geq f_2 \geq \cdots \geq f \geq 0\) be a sequence of measurable functions with range \([0, \infty]\) and \(\lim f_n = f\). Assume that \(f_1 \in L^1(\mu)\). Then prove that \(\lim \int_X f_n d\mu = \int_X f d\mu\). Give an example to show that the condition \(f_1 \in L^1(\mu)\) is necessary.

Let \(A \subset X\) be the set of points where \(f_1(x) = \infty\). Then \(\int_X f_1 d\mu = \int_A f_1 d\mu + \int_{A^c} f_1 d\mu\). But \(\int_A f_1 d\mu = \infty \cdot \mu(A)\) and the hypothesis \(f_1 \in L^1(\mu)\) implies \(\mu(A) = 0\). Now it is clear that we may replace \(X\) by \(A^c\), and thus we may assume that \(f_1 : X \to [0, \infty)\) (which of course implies this for all the \(f_n, f\)).

Consider the sequence \(g_n = f_1 - f_n\) (which makes sense, since these are functions to \([0, \infty)\)) and by hypothesis, \(0 \leq g_1 \leq g_2 \leq \cdots \leq f_1 - f\) and \(\lim g_n = f_1 - f\). So, by monotone convergence theorem, we have \(\lim \int_X g_n d\mu = \int_X (f_1 - f) d\mu = \int_X f_1 d\mu - \int_X f d\mu\). On the other hand, \(\int_X g_n d\mu = \int_X f_1 d\mu - \int_X f_n d\mu\) and since \(\int_X f_1 d\mu < \infty\), one can cancel and arrive at the desired result.

Let \(X = \mathbb{N}\) with the counting measure. Let \(f_n(m) = 0\) for \(m \leq n\) and 1 otherwise. These are measurable, since our \(\sigma\)-algebra is the power set of \(\mathbb{N}\). Then \(\lim f_n = 0\), but \(\int_X f_n d\mu = 1 \cdot \mu(\{m > n\}) = \infty\) for all \(n\). (is it possible to construct such examples with \(\mu(X) < \infty\)?)

(2) Let \(A\) be a measurable set and consider the sequence of functions, \(f_n = \chi_A\) if \(n\) is odd and \(f_n = 1 - \chi_A\) if \(n\) is even. Use this to deduce that strict inequality can occur in Fatou’s lemma.

One easily checks that \(\liminf f_n = 0\). Also one has, \(\int_X f_n d\mu = \mu(A)\) if \(n\) is odd and \(\int_X f_n d\mu = \mu(A^c)\) if \(n\) is even and thus \(\liminf \int_X f_n d\mu = \min\{\mu(A), \mu(A^c)\}\). Thus, if this minimum is non-zero, we get strict inequality in Fatou’s lemma.

(3) Suppose \(\mu(X) < \infty\) and \(f_n : X \to \mathbb{R}\) is a sequence of bounded (that is, \(|f_n| < M_n\) for constants \(M_n\)) measurable functions, converging uniformly to \(f\). Prove that \(\lim \int_X f_n d\mu = \int_X f d\mu\).

Since the convergence is uniform, we see that \(f_n\) is uniformly bounded. That is, there exists an \(M\) such that \(|f_n| < M\) for all \(n\). Thus \(f\) is bounded too by \(M\). Since \(\mu(X) < \infty\), the constant function \(M \in L^1(\mu)\). Now, apply Lebesgue Dominated convergence.

(4) Let \(f : X \to [0, \infty]\) be measurable and let \(A_1, A_2, \ldots\) be a countable collection of pairwise disjoint measurable sets and let \(A = \bigcup A_n\). Prove that \(\int_A f d\mu = \sum_{n=1}^{\infty} \int_{A_n} f d\mu\).
This is just Lebesgue monotone convergence. Let $B_n = \bigcup_{i=1}^{n} A_i$ and let $f_n(x) = f(x)$ if $x \in B_n$ and zero otherwise. Then it is easy to see that $f_n$ is measurable and $\lim f_n = f$ on $A$. Also, $0 \leq f_1 \leq f_2 \leq \cdots \leq f$. So, we can apply the theorem to get, $\lim \int_A f_n d\mu = \int_A f d\mu$. Since $f_n = 0$ outside $B_n$ and equal to $f$ on $B_n$, we have $\int_A f_n d\mu = \int_{B_n} f d\mu$. Writing $1 = \sum_{i=1}^{n} \chi_{A_i}$ on $B_n$, we see that, $\int_{B_n} f d\mu = \sum_{i=1}^{n} \int_{B_n} \chi_{A_i} f d\mu = \sum_{i=1}^{n} \int_{A_i} f d\mu$.

Taking limits, rest is clear.

(5) Let $f \in L^1(\mu)$. Prove that, given $\epsilon > 0$, there exists a $\delta > 0$ such that if $A \in \mathcal{F}$ with $\mu(A) < \delta$ then $\int_A |f| d\mu < \epsilon$.

Clearly we may replace $f$ by $|f|$ and thus assume that $f \geq 0$. Let $A_n = \{ x \in X \mid n-1 \leq f(x) < n \}$. Then, $A_n$ is measurable for all $n$, pairwise disjoint and $\cup A_n = X$. Thus, from the previous problem, we get,

$$\int_{B_n} f d\mu = \int_{B_n} \left( \sum_{i=1}^{n} \chi_{A_i} \right) f d\mu = \sum_{i=1}^{n} \int_{B_n} \chi_{A_i} f d\mu = \sum_{i=1}^{n} \int_{A_i} f d\mu$$

Since $f \geq n-1$ on $A_n$, we see that $\int_{A_n} f d\mu \geq (n-1)\mu(A_n)$. So, we see that the series $\sum_{n=1}^{\infty} (n-1)\mu(A_n)$ converges. Since the term $n = 1$ contributes nothing, we can say that $\sum_{n=2}^{\infty} (n-1)\mu(A_n)$ converges and since $n-1 \geq 1$ for all $n \geq 2$, we also get by comparison test, that $\sum_{n=2}^{\infty} \mu(A_n)$ converges. Adding, we get $\sum_{n=2}^{\infty} n\mu(A_n)$ converges.

Given $\epsilon > 0$, choose $N$ sufficiently large, so that $\sum_{n=N+1}^{\infty} n\mu(A_n) < \epsilon/2$. Notice that since $f < n$ on $A_n$, we have $\int_{G} f d\mu < \epsilon/2$ where $G = \cup_{n>\epsilon} A_n$. On $G^c$ we have $f < N$. So, choose $\delta = \frac{\epsilon}{2N}$. If $E \in \mathcal{F}$ with $\mu(E) < \delta$, consider $E_1 = E \cap G$, $E_2 = E \cap G^c$. Since $E_1 \subset G$, we have $\int_{E_1} f d\mu \leq \int_{G} f d\mu < \epsilon/2$. Since $E_2 \subset G^c$, we have $f < N$ on $E_2$ and thus, $\int_{E_2} f d\mu < \int_{E_2} Nd\mu = N\mu(E_2) < \epsilon/2$. Thus, $\int_{E} f d\mu < \epsilon$. Since $E \in \mathcal{F}$, we have $f < N$ on $E$ and thus, $\int_{E} f d\mu < \epsilon$. Therefore, $\int_{E} f d\mu < \epsilon$. So, we have $\int_{A_n} f d\mu < \epsilon$. Hence, $\int_A f d\mu < \epsilon$. Since $\epsilon > 0$ was arbitrary, we have $\int_A f d\mu < \epsilon$.
\[ \epsilon/2, \text{ since } \mu(E_2) \leq \mu(E) < \frac{\epsilon}{2N}. \] So,

\[ \int_E f \, d\mu = \int_{E_1} f \, d\mu + \int_{E_2} f \, d\mu < \epsilon. \]

(6) Let \( f_n : X \to \mathbb{R} \) be a sequence of measurable functions. Prove that the set of points where \( \{f_n\} \) converge is a measurable set.

For any \( p, n, m \in \mathbb{N} \), the set \( A(p, n, m) = \{ x \in X | |f_n(x) - f_m(x)| < \frac{1}{p} \} \) is clearly measurable. Thus for any \( N \in \mathbb{N} \), the set \( A(N, p) = \cap_{n, m \geq N} A(p, n, m) \) is measurable, being a countable intersection of measurable sets. Thus the set \( A(p) = \cup_{N=1}^{\infty} A(N, p) \) is measurable and then the set \( A = \cap_{p=1}^{\infty} A(p) \) is measurable. I claim that \( A \) is precisely the set of points where \( f_n \) converge.

First assume that \( f_n(x) \) converge. Then for any \( p \), we have an \( N \) such that \( |f_n(x) - f_m(x)| < \frac{1}{p} \) for all \( n, m \geq N \). So, we see that \( x \in A(N, p) \) and thus \( x \in A(p) \). Since this is true for any \( p, x \in A \).

Conversely, if \( x \in A \), given \( p \in \mathbb{N} \), we have \( x \in A(p) \) and thus \( x \in A(N, p) \) for some \( N \). One easily sees that this means \( \{f_n(x)\} \) is a Cauchy sequence.