(1) We define an abelian group $D$ to be \textit{divisible} if for any non-zero integer $n$, multiplication by $n$ on $D$ is surjective. In other words, given any $a \in D$, there exists $b \in D$ such that $nb = a$. For example, $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are both divisible and so is $\mathbb{C}^*$, non-zero complex numbers with respect to multiplication.

Let $D$ be a divisible group and let $A$ be a subgroup of an abelian group $B$. Given any homomorphism $f : A \rightarrow D$, show that there exists a homomorphism $g : B \rightarrow D$ such that $g(a) = f(a)$ for all $a \in A$. (If you are unfamiliar with transfinite induction, i.e. Zorn’s lemma, you may assume $B$ and thus $A$ are finitely generated abelian groups.) Deduce that if $D$ is a subgroup of an abelian group $G$, then $G$ has a subgroup $H$ such that the natural map $D \oplus H \rightarrow G$ is an isomorphism.

We look at all pairs $(C, \phi)$ where $C$ is a subgroup of $B$ containing $A$ and $\phi : C \rightarrow D$ is a homomorphism extending $f$, that is $\phi(a) = f(a)$ for all $a \in A$. This collection is non-empty, since $(A, f)$ belongs to it. Introduce a partial order on this collection by saying $(C, \phi) \leq (C', \phi')$ if $C \subset C'$ and $\phi'$ extends $\phi$. If $(C_1, \phi_1) \leq (C_2, \phi_2) \leq \cdots$ is a totally ordered collection (for simplicity, I have indexed it with $\mathbb{N}$, but you should have no difficulty in doing this with an arbitrary ordered indexing set) then consider $C = \bigcup C_i$, a subgroup of $B$ containing $A$ and define $\phi(c) = \phi_i(c)$ for any $c \in C_i$. By our ordering, this map is well defined and thus $(C, \phi)$ is a maximal element for the above sequence of groups with respect to the partial order. So, every totally ordered set has a maximal element and thus by Zorn’s lemma, our collection has a maximal element, say $(G, g)$.

If $G = B$, we are done. So assume not and let $a \in B$, $a \notin G$ and let $G'$ be the subgroup generated by $G, a$ and let $P$ be the cyclic subgroup generated by $a$. If $P \cap G = \{0\}$, one easily checks that $G' = G \oplus P$ and then we can define $h : G' \rightarrow D$ by $h((b, c)) = g(b)$ and it has all the required properties, contradicting maximality of $(G, g)$. So, assume $P \cap G \neq 0$. Since subgroup of a cyclic subgroup is cyclic, we see that $P \cap G$ is the cyclic group generated by $na$ for some $n \geq 2$. Since $na \in G$, we can find an element $d \in D$, by divisibility such that $nd = g(na)$. Define a map on $G'$ by sending $G$ via $g$ and $a$ to $d$. One easily checks that this map extends $g$, again contradicting maximality.
For the last part, if $D \subset G$, by the first part, we can extend the identity map $D \to D$ to a map $\phi : G \to D$. Easy to see that $G = D \oplus \ker \phi$.

(2) Let $I(p)$ for a prime $p$ denote the set of all elements $a$ in $\mathbb{Q}/\mathbb{Z}$ such that $p^n a = 0$ for some $n$. Show that $I(p)$ is a divisible subgroup and $\mathbb{Q}/\mathbb{Z} = \bigoplus_p I(p)$.

Both parts are straight forward.

(3) Show that the natural map

$$\text{Hom}(\bigoplus_{i=1}^{n} G_i, \bigoplus_{j=1}^{m} H_j) \to \bigoplus_{i,j} \text{Hom}(G_i, H_j)$$

is an isomorphism of groups, where $G_i, H_j$ are abelian groups.

This is clear too.

(4) Show that for any positive integer $n$, there exists a unique cyclic subgroup of order $n$ in $\mathbb{C}^*$. It is the cyclic group generated by $z = \exp(2\pi i/n)$.

(5) For a finite abelian group $G$ define its dual $G^\vee$ to be the group $\text{Hom}(G, \mathbb{C}^*)$. Show that $o(G) = o(G^\vee)$.

Writing $G$ as a direct sum of cyclic groups, $\bigoplus_{i=1}^{n} C_i$ and using an earlier exercise, we have $\text{Hom}(G, \mathbb{C}^*) = \bigoplus \text{Hom}(C_i, \mathbb{C}^*)$. So, it is immediate that we need to prove the statement only for cyclic groups. So, let $C$ be a cyclic group of order $n$ and let $C'$ the unique cyclic subgroup of order $n$ in $\mathbb{C}^*$. Then $\text{Hom}(C, \mathbb{C}^*) = \text{Hom}(C, C')$ and the latter has order $n$ is clear.

(6) Show that the natural map $G \to G^\vee\vee$ given below is an isomorphism. For any $g \in G$, we get an element of $G^\vee\vee$ as follows. Define a map $G^\vee \to \mathbb{C}^*$ by for $\phi \in G^\vee$ maps to $\phi(g)$.

This is just a simple checking, and can be first reduced as before to the cyclic group situation.