Homework 8

(1) Let $A$ be an integral domain. Describe all ring automorphisms
\[ \phi : A[X] \rightarrow A[X] \] and \[ \psi : A[[x]] \rightarrow A[[x]] \] with $\phi(a) = a, \psi(a) = a$ for all $a \in A$.

(2) Let $M$ be a module over $R = k[[x_1, \ldots, x_n]], k$ a field, which
is complete with respect to the maximal ideal $m$ generated by
the $x_i$ s. Assume that $M/mM$ is a finite dimensional vector
space over $R/m = k$. Show that $M$ is finitely generated over $R$. (One can easily deduce Weierstrass preparation theorem for
power series rings over a field from this.)

(3) Let $R$ be an UFD and $K$ its fraction field. Show that if $x \in K$
such that $x^n + a_1x^{n-1} + \cdots + a_n = 0$ with $a_i \in R$, then $x \in R$.
(This is usually described as UFDs are integrally closed).

(4) Let $R = \mathbb{C}[x, y], S = \mathbb{C}[[x, y]]$ and $F = y^2 - x^3, G = xy + (x+y)^3$.
(a) Show that $F, G$ are irreducible in $R$.
(b) Show that neither $R/FR, R/GR$ is a UFD.
(c) Show that $F$ is irreducible but $G$ is not in $S$. 