Homework 3

(1) Let $A$ be an integral domain. Show that $A$ is integrally closed if and only if $A_P$ (localization at the prime ideal $P$) is integrally closed for all primes $P$.

(2) If $P$ is a prime ideal in a ring $A$, show that $\dim A_P \leq \dim A$.

(3) We say that a local domain $A$ satisfies Serre condition $S_2$, if there exists two non-zero elements $x, y$ in the maximal ideal such that if $ax = by$ for some $a, b \in A$, then $a = py, b = px$ for some $p \in A$. Show that if $x, y$ is as above, and $ax^n = by^m$ for some $n, m \in \mathbb{N}$, then $a = py^m, b = px^n$ for some $p \in A$.

(4) A Noetherian integral domain of dimension one, which is integrally closed is called a Dedekind domain. We assume below that $A$ is a Dedekind domain and $K$ its fraction field. Also assume the following (which we shall prove later): For any $0 \neq a \in A$, there are only finitely many prime ideals containing $a$.

(a) If $0 \neq I \subset K$ is a finitely generated $A$-module, define $I^{-1} = \{x \in K | xI \subset A\}$. Show that $II^{-1} = A$. (The notation $II^{-1}$ as usual denotes the set of all finite sums of elements $ab$ with $a \in I, b \in I^{-1}$). Deduce that $I^{-1}$ is a finitely generated $A$-module. We call such a non-zero finitely generated $A$-submodule of $K$, a fractionary ideal.

(b) Show that the set of fractionary ideals form an abelian group with multiplication as above.

(c) If $I$ is any fractionary ideal, show that there exists prime ideals $P_1, \ldots, P_n$ and $0 \neq e_i \in \mathbb{Z}$ (both unique) such that $I = P_1^{e_1} \cdots P_n^{e_n}$.

(5) Let $A$ be a Noetherian integral domain of finite Krull dimension. Then $A$ is integrally closed if and only if the following conditions are satisfied.

(a) $A_P$ is integrally closed for any prime ideal with $\dim A_P = 1$.

(b) If $\dim A_P > 1$, then $A_P$ satisfies $S_2$ condition above.

This is called Serre’s criterion for integrally closed rings.