Homework 5

(1) Let $0 \to F_1 \to F_0 \to M \to 0$ be an exact sequence of $A$-modules for a commutative ring $A$. Recall that this means $F_0 \to M$ is onto, image of $F_1 \to F_0$ is equal to the kernel of $F_0 \to M$ and $F_1 \to F_0$ is injective. Show that for any multiplicative closed subset $S$, one has exact sequences, $0 \to S^{-1}F_1 \to S^{-1}F_0 \to S^{-1}M \to 0$ and for any ideal $I$, $F_1/IF_1 \to F_0/IF_0 \to M/IM \to 0$. For the latter you do not need to assume that $F_1 \to F_0$ is injective.

The localization part we checked in class. All of these are routine checking. So, let me check the second one.

Since given any element $\overline{m} \in M/IM$, we can find $m \in M$ such that $m \mapsto \overline{m}$ and there exists some $e \in F_0$ such that $e \mapsto m$ by surjectivity of $F_0 \to M$, it is easy to check that the image of $e$ in $F_0/IF_0$ will map to $\overline{m}$, proving surjectivity of $F_0/IF_0 \to M/IM$.

If $\overline{e} \in F_0/IF_0$ goes to zero in $M/IM$ where $\overline{e}$ is the image of $e \in F_0$, it just means that image of $e$ is in $IM$. So write image of $e = \sum a_i m_i$, where $a_i \in I, m_i \in M$. Then, by surjectivity of $F_0 \to M$, we can find $e_i \in F_0$ so that $e_i \mapsto m_i$. Since we can replace $e$ by $e - \sum a_i e_i$, since both go to $\overline{e}$, we see that we can further assume that $e$ actually maps to zero in $M$. But then, exactness says, we can find $f \in F_1$ which maps to $e$. The rest is easy.

(2) Let $A$ be a Noetherian ring and $M$ a finitely generated module over $A$. Define the support of $M$ as

$$\text{Supp } M = \{P | P \text{ prime ideal, } M_P \neq 0\}.$$  

We have defined in class the annihilator of $M$, $\text{Ann } M = \{a \in A | aM = 0\}$, minimal primes and associated primes of $M$.

(a) Show that $\text{Ass } M \subset \text{Supp } M$.

Let $P$ be an associated prime. Then we have an inclusion $A/P \subset M$ and as said in the previous part, we get $(A/P)_P \subset M_P$. But, $(A/P)_P$ is just the fraction field of $A/P$ and in particular not zero. So, $M_P \neq 0$.

(b) If $N \subset M$ is a submodule, show that $\text{Supp } M = \text{Supp } N \cup \text{Supp } (M/N)$.

Just use that for any prime ideal $P$ we have $N_P \subset M_P$ and $(M/N)_P = M_P/N_P$.

(c) Show that $P \in \text{Supp } M$ if and only if $\text{Ann } M \subset P$.
Assume \( P \in \text{Supp} \, M \) and \( P \) does not contain \( \text{Ann} \, M \).
Then we can pick \( a \in \text{Ann} \, M \), \( a \not\in P \). Thus, when we localize at \( P \), any element of \( M_P \) can be written as (the equivalence class of) \( (am, a) = (0, a) = 0 \). So, \( M_P = 0 \).
Conversely, assume that \( P \) contains the annihilator. If \( M_P = 0 \), then any element \( m \in M \) becomes zero in \( M_P \) and thus there exists an \( a \not\in P \) such that \( am = 0 \). Since \( M \) is finitely generated, we can find a single such \( a \) such that it annihilates all these finite set of generators and thus all of \( M \). Again a contradiction.

(3) Let \( A, M \) be as above and let \( P \) be a maximal ideal and let \( r \) be the dimension of \( M/PM \) as an \( A/P \) vector space. Show that there exists an \( f \not\in P \) such that for all maximal ideals \( Q \) not containing \( f \), the dimension of \( M/QM \) as \( A/Q \) vector space is at most \( r \).

Since \( A \) is Noetherian and \( M \) is finitely generated, it is finitely presented and thus we can find an exact sequence \( F_1 \rightarrow F_0 \rightarrow M \rightarrow 0 \), where \( F_i \) s are free modules of finite rank. So, the map \( F_1 \rightarrow F_0 \) is given by a matrix \( \alpha \) over \( A \) of size \( m \times n \), \( m, n \) being the ranks of \( F_0, F_1 \). The fact that \( \dim_{A/Q} M/QM = s \) can then be seen to be equivalent to the rank of \( \alpha \) modulo \( Q \) is precisely \( m - s \), by using the exactness of \( F_1/QF_1 \rightarrow F_0/QF_0 \rightarrow M/QM \rightarrow 0 \). Thus at \( P \) rank of \( \alpha \) is \( m - r \). So, some \( m - r \times m - r \) minor is not in \( P \). So, let \( f \) be this minor. Then for any \( Q \) not containing this minor, clearly the rank of \( \alpha \) is at least \( m - r \) or equivalently, \( \dim_{A/Q} M/QM \leq r \).

(4) Let \( A = k[x_1, \ldots, x_r]/I \) where \( k \) is a field, the maximal ideal generated by the all the variables has some power contained in \( I \). Thus, \( A \) is an Artin local ring with the maximal \( P \) generated by the images of the variables, which by abuse of notation, we will still call \( x_i \). Let \( M \) be a finitely generated module over \( A \).
(a) If \( \dim_k M = d \), show that for any chain of submodules \( M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_p \) of \( M \), \( p \leq d \) and there is such a chain with \( p = d \). Further show that for such a chain with \( p = d \), \( M_i/M_{i-1} \cong k = A/P \) for \( i > 0 \). As we defined in class, \( d \) is called the length of \( M \) and denoted by \( \ell(M) \).
Proof is by induction on \( d \). If \( d = 0 \), \( M = 0 \). Let us look at the case of \( d = 1 \). Then \( M \) is generated by a single element and so \( M = A/J \). Since \( J \neq A \), \( J \subset P \) and so \( A/J \) maps onto \( A/P = k \). Since \( \dim_k A/J = \dim_k A/P = 1 \), we see that \( J = P \) and then the argument is trivial.
Assume result proved for $d - 1$. For $d > 0$, $M/PM \neq 0$ by Nakayama’s lemma and then $M/PM$ maps onto $A/P = k$, by vector space arguments. Thus we have a surjection $M \to k$ and let $N$ be its kernel. Then $\dim_k N = d - 1$ and an easy induction finishes the proof.

(b) If $N \subset M$, show that $\ell(M) = \ell(N) + \ell(M/N)$.
This is just the corresponding statement for vector spaces.

(c) Let $\omega_A = \text{Hom}_k(A,k)$, the dualizing module. Show that the map $T : \text{Hom}_A(M,\omega_A) \to \text{Hom}_k(M,k)$ defined as, $T(f)(m) = f(m)(1)$ is an $A$-module isomorphism.

It is clear that $T$ is a $k$-linear map. Let $a \in A$ and we will show that $T(af) = aT(f)$. $T(af)(m) = (af)(m)(1) = f(m)(a)$. $aT(f)(m) = a(f(m)(1)) = f(m)(a)$. Next we check that the map is injective. If $T(f) = 0$, then for any $m \in M$, we have $T(f)(m) = 0$ which means $f(m)(1) = 0$. But, then $f(m)(a) = f(am)(1) = 0$ and thus $f(m) = 0$ for all $m$, which means $f = 0$. The inverse to $T$ is defined as $U : \text{Hom}_k(M,k) \to \text{Hom}_A(M,\omega_A)$ by $U(g)(m)(a)$ to be (where $m \in M, a \in A$) $g(am)$. I will leave you to check that this is an inverse.

(d) Show that $\ell(\text{Hom}_A(M,\omega_A)) = \ell(M)$. Deduce that the natural map $M \to \text{Hom}_A(\text{Hom}_A(M,\omega_A),\omega_A)$ is an isomorphism.
Trivial from the above.