(1) We say a commutative ring $A$ is graded, if it can be written as $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$, where $A_0$ is a ring, $A_i$s are $A_0$ modules and the addition in $A$ is the obvious one and $A_i A_j \subset A_{i+j}$ for all $i, j$.

(a) Show that $A = k[x_1, \ldots, x_n]$, polynomial ring in $n$ variables over a field $k$ is a graded ring where $A_d$ is the $k$ vector space of all homogeneous polynomials of degree $d$. This is obvious, since the product of a homogeneous polynomial of degree $d$ and another of degree $e$ has the product homogeneous of degree $d+e$.

(b) Let $A$ be a graded ring and let $D : A \to A$ be the map defined as $D((a_0, a_1, a_2, \ldots, a_n, \ldots)) = (0, a_1, a_2, \ldots, na_n, \ldots)$ is a derivation. It is easy to show that $D(f + g) = D(f) + D(g)$. Thus using distributivity, it suffice to show Leibniz’ rule for $f \in A_i, g \in A_j$. Then $fg \in A_{i+j}$ and so $D(fg) = (i+j)fg$. On the other hand $fD(g) + gD(f) = f \cdot jg + g \cdot if = (i+j)fg$.

(2) A few problems on derivations in positive characteristic $p > 0$.

All rings and fields will be of characteristic $p$.

(a) Let $K \subset K(x) = L_1$ with $x \notin K$ and $x^p \in K$, be a field extension. Show that $D = \text{Der}_K(L_1)$ is a one dimensional $L_1$-vector space generated by $D = \frac{d}{dx}$. Calculate $(xD)^p$.

First, we show that $D \in D$. Any element in $L_1$ can be uniquely written as $f = a_0 + a_1 x + \cdots + a_{p-1} x^{p-1}$ with $a_i \in K$ and so we get $K$-linear map $D(f) = a_1 + 2a_2 x + \cdots + (p-1)a_{p-1} x^{p-2}$. To check that it is a derivation, suffices to show that $D(x^n) = nx^{n-1}$ and this is obvious. Let $T \in D$ and let $f = T(x)$. Then $S = T - fD \in D$ and $S(x) = 0$. So, we get $S(x^n) = 0$ for any $n$ and the rest is clear.

Easy to check that $(xD)^p = xD$.

(b) Prove a similar result for $L_n = K(x_1, \ldots, x_n)$, where $x_i^p \in K$, by first showing that we may choose $n$ such that $p^n = [L_n : K]$. Again, calculate $E^p$, where $E$ is the Euler derivation, $E = \sum x_i \frac{d}{dx_i}$. This is very similar to the previous problem.

(3) A few problems on modules and functors which often appear.

Let $A$ be a commutative ring and let $C$ be the category of $A$-modules.
(a) Let \( X \in \mathcal{C} \) and define a map \( F : \mathcal{C} \to \mathcal{C} \) by \( M \mapsto \text{Hom}_A(X, M) \). Show that this defines a functor.

Notice that \( \text{Hom}_A(X, M) \) is an \( A \)-module and so \( F \) defines a map from the objects of \( \mathcal{C} \) to itself. Next, given a homomorphism of \( A \)-modules, \( f : M \to N \), we get a natural homomorphism \( F(M) \to F(N) \), by mapping \( \phi \in F(M) \) to \( f \circ \phi \in F(N) \). Rest is just checking.

(b) If \( 0 \to M \to N \to P \) is an exact sequence in \( \mathcal{C} \), show that the sequence, \( 0 \to F(M) \to F(N) \to F(P) \) is exact. Such functors are called left exact. Further show by an example that, even if we had surjectivity from \( N \to P \) above, \( F(N) \to F(P) \) may not be surjective.

The exactness is easy. I will write an example to show the last part. Let \( A = \mathbb{Z} \) and consider an exact sequence \( 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0 \), where \( n \neq 0 \) is an integer. Let \( X = \mathbb{Z}/n\mathbb{Z} \). Then, \( \text{Hom}_\mathbb{Z}(X, \mathbb{Z}) = 0 \), while \( \text{Hom}_\mathbb{Z}(X, \mathbb{Z}/n\mathbb{Z}) \neq 0 \).

(c) Similarly, consider the map \( G : \mathcal{C} \to \mathcal{C} \) given by \( M \mapsto \text{Hom}_A(M, X) \). Show that this too is a functor (to be precise, to the opposed category, since it will reverse arrows).

As before, show that if we have an exact sequence \( M \to N \to P \to 0 \), we get an exact sequence \( 0 \to G(P) \to G(N) \to G(M) \).

Less said the better.