(1) Let $V$ be a vector space of dimension 2, with basis $e_1, e_2$ over a field $k$. Show that $V \otimes_k V$ is a vector space with basis $e_i \otimes e_j, 1 \leq i, j \leq 2$. If $\sum a_{ij} e_i \otimes e_j \in V \otimes V$ is decomposable, show that $a_{11}a_{22} = a_{12}a_{21}$ and thus not all elements of $V \otimes V$ are decomposable.

We have done the first part in class, even more generally, by showing that $M \otimes (N \oplus P) = M \otimes N \oplus M \otimes P$ for any three modules $M, N, P$ over a ring $A$.

The second part is equally easy. A decomposable vector is of the form $(ae_1 + be_2) \otimes (ce_1 + de_2)$. Then we see that (by multiplying out) $a_{11} = ac, a_{12} = ad, a_{21} = bc$ and $a_{22} = bd$.

(2) Let $K \subset L$ be a finite extension of fields. Show that this extension is separable if and only if the ring $L \otimes_K K$ has no non-trivial nilpotent elements.

Assume that $K \subset L$ is separable and assume that $L \otimes_K K$ has a non-trivial nilpotent $\alpha = \sum_{i=1}^n a_i \otimes b_i, a_i \in L, b_i \in K$. Let $M = K(a_1, \ldots, a_n) \subset L$. Then, we have $M \otimes_K K \subset L \otimes_K K$, since everything is flat over $K$. So, $\alpha \in M \otimes_K K$ and it is a non-trivial nilpotent in this ring. So, we may replace $L$ by $M$ and assume that $M$ is finite and separable over $K$. Then it is simple, so we can write it as $K[X]/(p(X))$ for some irreducible separable polynomial over $K$. Then, $M \otimes_K K = K[X]/(p(X))$. Since $p$ is separable, we have $p(X) = \prod (X - \lambda_i)$ with $\lambda_i \in K$ distinct. So, this ring is $\prod K[X]/((X - \lambda_i)) = \prod K$ and in particular have non non-trivial nilpotents.

Conversely, assume that $L$ is not separable. Then characteristic is $p > 0$ and we can assume that there exists an $\alpha \in L - K$ with $\alpha^p = u \in K$. Then, as before, we consider $M = K(\alpha) = K[X]/(X^p - u)$ and $M \otimes_K K \subset L \otimes_K K$. But, the former is just $K[X]/((X - \alpha)^p)$ and we have a non-trivial nilpotent, namely $X - \alpha$.

(3) Let $\phi : A \to B$ be a ring homomorphism (as usual, rings are commutative with 1). Let $M$ be an $A$-module and $N$ be a $B$-module. So, $N$ is also an $A$-module, using $\phi$.

(a) Show that $M \otimes_A B$ is a $B$-module via the action, $b(m \otimes c) = m \otimes bc$, for $m \in M, b, c \in B$.

(b) Show that $\text{Hom}_A(M, N)$ is a $B$-module via the action $(bf)(m) = bf(m)$ for $b \in B, f \in \text{Hom}_A(M, N)$.
(c) Show that $\text{Hom}_B(M \otimes_A B, N)$ and $\text{Hom}_A(M, N)$ are naturally isomorphic as $B$-modules.

These are direct checking and easy. I will just comment on the last one. Given $f \in \text{Hom}_A(M, N)$, we get an $A$-bilinear map $f' : M \times B \to N$, by $f'(m, b) = bf(m)$. So, we get a unique $A$-module homomorphism $g : M \otimes_A B \to N$, which one easily checks to be a $B$-module homomorphism. This defines a map $\text{Hom}_A(M, N) \to \text{Hom}_B(M \otimes_A B, N)$ and one checks that this is a $B$-module isomorphism.

(4) Let $A \to B$ be a ring homomorphism (as usual, rings are commutative with 1).

(a) Show that there exists a natural surjective ring homomorphism $\phi : B \otimes_A B \to B$ with $\phi(x \otimes y) = xy$.

(b) Show that the kernel of $\phi = I$ is generated as a $B \otimes_A B$ ideal by elements of the form $x \otimes 1 - 1 \otimes x, x \in B$.

It is clear that these elements are in the kernel. It is trivial to show that the kernel is generated by elements of the form $b \otimes c - 1 \otimes bc$. But, this is just $(1 \otimes c)(b \otimes 1 - 1 \otimes b)$.

(c) Show that (easy) $I/I^2$ is a $B$ module and there exists a well defined map $d : B \to I/I^2$, with $d(x) = x \otimes 1 - 1 \otimes x \in I/I^2$.

This is clear.

(d) Show that $d$ is an $A$-derivation.

First, $d$ is additive, since $d(x + y) = d(x) + d(y)$, almost by definition. If $a \in A, d(a) = a \otimes 1 - 1 \otimes a$. But $a \otimes b = 1 \otimes ab$ for $a \in A, b \in B$ and so $d(a) = 0$. Finally we calculate $d(xy)$. This is $xy \otimes 1 - 1 \otimes xy$. We write $xy \otimes 1 = xy \otimes 1 - x \otimes y + x \otimes y$ which is just $(x \otimes 1)(y \otimes 1 - 1 \otimes y) + x \otimes y = xd(y) + x \otimes y$. So, we only need to calculate $x \otimes y - 1 \otimes xy$ and this is just $(1 \otimes y)(x \otimes 1 - 1 \otimes x) = yd(x)$.

(e) If $D : B \to M$ is any $A$-derivation, where $M$ is a $B$-module, show that there exists a $B$-module homomorphism $f : I/I^2 \to M$ such that $D = f \circ d$.

($I/I^2$ is called the module of Kähler differentials and denoted by $\Omega^1_{B/A}$.)

Since $D$ is an $A$-module homomorphism (it is NOT a $B$-module map), we get a map $D \otimes Id : B \otimes_A B \to M \otimes_A B$ and we also have a natural map $M \otimes_A B \to M$, since $M$ is a $B$-module. Thus we get a map $B \otimes_A B \to M$ and thus a map $I \to M$. One checks easily that this factors through
$I/I^2$ and then we get a $B$-module map $f : I/I^2 \to M$ having the required properties.

(5) If $M$ is an $A$-module and $I \subset A$ is an ideal, show that $M \otimes_A A/I$ is naturally an $A/I$-module and isomorphic to $M/IM$.

This easily follows from the exact sequence $0 \to I \to A \to A/I \to 0$ and tensoring with $M$, we get an exact sequence, $I \otimes M \to A \otimes M = M \to A/I \otimes M \to 0$. The image of $I \otimes M \to M$ is just $IM$ and thus $M/IM = A/I \otimes M$. 