• Given a 2-variable function $f(x,y)$, define

$$\frac{\partial f}{\partial x}(x,y) := \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}, \quad \frac{\partial f}{\partial y}(x,y) := \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}.$$  

We also use the notation $f_x(x,y), f_y(x,y)$.

• These represent slopes in the $x$- and $y$-directions. More precisely, if you slice the graph of $z = f(x,y)$ by the plane $y = y_0$, you get a curve (the graph of $z = f(x,y_0)$) in this plane. The slope of the tangent line to this curve at $(x_0, y_0, f(x_0, y_0))$ is $\frac{\partial f}{\partial x}(x_0, y_0)$. This tangent line is parametrized by $t \mapsto (x_0 + t, y_0, f(x_0, y_0) + f_x(x_0, y_0)t)$.

• Stupid examples: if $f(x,y) = x^a y^b$, then $\frac{\partial f}{\partial x} = ax^{a-1}y^b, \frac{\partial f}{\partial y} = bx^a y^{b-1}$. If $f(x,y) = e^x$, then $\frac{\partial f}{\partial x} = e^x, \frac{\partial f}{\partial y} = 0$. When taking $\frac{\partial}{\partial x}$, you have to view $y$ as a constant, which is consistent with the geometric meaning of the partial derivative just described.

• Just as with ordinary derivatives, you may iterate partial derivatives: $f_{xx} = \frac{\partial^2 f}{\partial x \partial x}, f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, f_{yx} = \frac{\partial^2 f}{\partial y \partial x}, f_{yy} = \frac{\partial^2 f}{\partial y \partial y}$.

• Clairaut’s theorem: if both $f_{xy}$ and $f_{yx}$ are continuous, then they are equal – i.e. the order in which you take partial derivatives doesn’t matter.

**Partial differential equations.**

• of fundamental importance in mathematical physics, finance, etc.

• Laplace equation: $f_{xx} + f_{yy} = 0$. Solutions are called *harmonic* functions (e.g. voltage in the absence of a potential field).

• Heat equation: $f_t = \alpha^2 f_{xx}$. (Here $f$ is a function of time $t$ and position $x$.) Rate of change of (say) temperature is proportional to its concavity at a point.
• Wave equation: $f_{tt} = a^2 f_{xx}$. Satisfied by propagating waves, as the name would imply!

• many other famous examples (Navier-Stokes, Black-Scholes, Schrödinger, not to mention the plethora of such equations in general relativity and quantum field theory..."