Dot product.

- If \( \vec{a} = \langle a_1, a_2, a_3 \rangle \) and \( \vec{b} = \langle b_1, b_2, b_3 \rangle \), then \( \vec{a} \cdot \vec{b} := a_1 b_1 + a_2 b_2 + a_3 b_3 \). This is a number (scalar). Notice that \( \vec{a} \cdot \vec{a} = \| \vec{a} \|^2 \), and that dot product satisfies commutative and distributive laws.

- Geometric interpretation: \( \vec{a} \cdot \vec{b} = \| \vec{a} \| \| \vec{b} \| \cos \theta \), where \( \theta \) is the angle between \( \vec{a} \) and \( \vec{b} \). (In class, I explained one way to check this.) In particular, \( \vec{a} \cdot \vec{b} = 0 \iff \vec{a} \perp \vec{b} \). More generally, this formula gives you a way to find \( \theta \).

- “Direction angles” are the angles \( \alpha, \beta, \gamma \) that \( \vec{a} \) makes with the standard basis vectors \( \hat{i} = \langle 1, 0, 0 \rangle \), \( \hat{j} = \langle 0, 1, 0 \rangle \), \( \hat{k} = \langle 0, 0, 1 \rangle \). The “direction cosines” are (you guessed it) \( \cos \alpha, \cos \beta, \cos \gamma \). The sum of their squares is 1 (why?).

- The “vector projection of \( \vec{b} \) onto \( \vec{a} \)” is the vector \( \text{proj}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{\| \vec{a} \|^2} \vec{a} \), and its length \( \text{comp}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{\| \vec{a} \|} \) is called the “scalar projection” or “component of \( \vec{b} \) along \( \vec{a} \)”: 

\[ \text{proj}_{\vec{a}} \vec{b} \]

- In physics, work is computed by taking the dot product \( W = \vec{F} \cdot \vec{D} \) of force and distance, assuming the force is constant over that distance. (Distance here is a vector pointing from start to finish.)
Cross-product.

- In the same notation as above (for $\vec{a}$ and $\vec{b}$),

$$\vec{a} \times \vec{b} := \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$ 

Unlike the dot product, this is a vector.

- Since $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$, $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$ and $\vec{b}$, in the direction according to the right-hand rule:

- The length

$$\| \vec{a} \times \vec{b} \| = \| \vec{a} \| \| \vec{b} \| | \sin \theta |$$

depends on the angle between $\vec{a}$ and $\vec{b}$. This is biggest when they are perpendicular, and zero when they are parallel.
Quick proof of equation (1). This is better than Stewart’s if you like summation notation. (If you don’t, read Stewart’s, or skip it.) In the sums, $i$ and $j$ run from 1 to 3.

$$\|\vec{a} \times \vec{b}\|^2 + (\vec{a} \cdot \vec{b})^2 = (\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b}) + \left( \sum_i a_i b_i \right)^2$$

$$= \sum_{i<j} (a_i b_j - a_j b_i)^2 + \left( \sum_i a_i^2 b_i^2 + \sum_{i \neq j} a_i b_i a_j b_j \right)$$

$$= \sum_{i<j} a_i^2 b_j^2 + \sum_{i<j} a_j^2 b_i^2 - 2 \sum_{i<j} a_i b_j a_j b_i + \left( \sum_i a_i^2 b_i^2 + 2 \sum_{i<j} a_i b_i a_j b_j \right)$$

$$= \left( \sum_i a_i^2 \right) \left( \sum_j b_j^2 \right)$$

$$= \|\vec{a}\|^2 \|\vec{b}\|^2.$$  

Since $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos \theta$, we have

$$\|\vec{a} \times \vec{b}\|^2 = \|\vec{a}\|^2 \|\vec{b}\|^2 (1 - \cos^2 \theta) = \|\vec{a}\|^2 \|\vec{b}\|^2 \sin^2 \theta,$$

and taking square roots on both sides gives (1).