• Consider a vector field \( \vec{F}(x, y) = P(x, y)\hat{i} + Q(x, y)\hat{j} \) on a region (connected open set) \( D \) in \( \mathbb{R}^2 \). If \( \vec{F} = \nabla f \) — that is, if \( \vec{F} \) is conservative — then \( \int_C \vec{F} \cdot d\vec{r} \) only depends on the endpoints \( A \) and \( B \) of \( C \) (independence of path). In particular, if \( C \) is closed (\( A = B \)), then \( \int_C \vec{F} \cdot d\vec{r} = 0 \).

• Moreover, if \( \vec{F} \) is conservative, then \( P = f_x, Q = f_y \), and so by Clairaut’s theorem, \( P_y = f_{xy} \) and \( Q_x = f_{yx} \).

• To state a converse to this last result, suppose \( D \) is simply connected: this means that it has no holes. Then \( P_y = Q_x \) implies that \( \vec{F} \) is conservative. We will check this if \( D \) is a rectangle.

• If \( D \) has a hole, then it is possible to have \( P_y = Q_x \) and \( \vec{F} \) still fail to be conservative.

• For instance, \( \vec{F} = -\frac{y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j} \) satisfies \( P_y = \frac{x^2-y^2}{(x^2+y^2)^2} = Q_x \), but if \( C \) is the (closed) circle of radius 1, then \( \int_C \vec{F} \cdot d\vec{r} = 2\pi \neq 0 \), so \( \vec{F} \) can’t be conservative. The problem is that the domain of definition of \( \vec{F} \) omits the origin, hence has a hole. We are saying there can’t be a function \( f \) on \( D = \mathbb{R}^2 - \{0\} \) such that \( \vec{F} = \nabla f \) on all of \( D \). If we take a smaller region \( D' \) inside \( D \) which doesn’t have a hole, like a disk of radius 1 about \( (2, 0) \), then the restriction of \( \vec{F} \) to \( D' \) is indeed conservative (and obviously \( D' \) doesn’t contain \( C \), so there is no contradiction).

• For a vector field \( \vec{F} \) on any region \( D \), \( \vec{F} \) is conservative if and only if \( \int_C \vec{F} \cdot d\vec{r} \) is independent of path (or equivalently, zero on all closed loops).

• We will discuss in lecture how to actually find \( f \) in the event that \( \vec{F} \) is conservative.