More on curl and div.

• Though the title says “plane”, we’ll start with some stuff in space. Recall that for a vector field $\vec{F}$ on a region $D \subset \mathbb{R}^3$, $\text{div}\vec{F} = \vec{\nabla} \cdot \vec{F}$ and $\text{curl}\vec{F} = \vec{\nabla} \times \vec{F}$, where $\vec{\nabla} = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k}$.

• By Clairaut’s Theorem (details in class), we have two identities: (1) $\text{curl}(\vec{\nabla} f) = \vec{0}$ for any function $f$ on $D$; and (2) $\text{div}(\text{curl}\vec{F}) = 0$ for any vector field $\vec{F}$ on $D$.

• By (1), if $\vec{F} = \vec{\nabla} f$ (i.e. $\vec{F}$ conservative), then $\text{curl}\vec{F} = \vec{0}$ (i.e. $\vec{F}$ irrotational). If $D$ is simply connected, then the converse holds: $\vec{F}$ irrotational $\implies \vec{F}$ conservative.

• By (2), if $\vec{F}$ is the curl of another vector field $\vec{G}$ (i.e. $\vec{F}$ is a “curl field”), then $\text{div}\vec{F} = 0$ (i.e. $\vec{F}$ incompressible). If $D$ has no “solid holes”, then the converse holds here too.

Vector forms of Green’s Theorem.

• Let $C$ be a simple closed curve in $\mathbb{R}^2$, with a smooth parametrization $\vec{r}(s) = x(s)\hat{i} + y(s)\hat{j}$ by arclength $s$, and “positively oriented” (i.e. in the counterclockwise direction). The unit tangent vector is $\hat{T}(s) = x'(s)\hat{i} + y'(s)\hat{j}$, and the outward-pointing unit normal is $\hat{n}(s) = y'(s)\hat{i} - x'(s)\hat{j}$.

• Now suppose $C = \partial S$, and that $D \subset \mathbb{R}^2$ contains $C$ and $S$. Let $\vec{F} = P\hat{i} + Q\hat{j}$ be a vector field on $D$ ($D$ contains $C$). Write

$$\int_C \vec{F} \cdot \hat{n} ds = \int_C (P\hat{i} + Q\hat{j}) \cdot (y'(s)\hat{i} - x'(s)\hat{j}) ds$$

$$= \int_C -Qx'(s) ds + Py'(s) ds = \int_C -Qdx + Pdy,$$
which by Green’s Theorem
\[
\oint_S (P_x - (-Q_y))\,dA = \iint_S (P_x + Q_y)\,dA.
\]
This gives Gauss’s Divergence Theorem in the plane:
\[
\oint_{\partial S} \vec{F} \cdot \hat{n} \,ds = \iint_S \text{div}(\vec{F})\,dA,
\]
which tells us that the total flux of $\vec{F}$ across the boundary $\partial S$ (the left-hand side) equals the integral of the “outward flux per unit area” over $S$, which sounds completely plausible.

• There is also “Stokes’s Theorem in the plane” which is more or less a restatement of Green’s theorem: it reads
\[
\oint_{\partial S} \vec{F} \cdot \hat{T} \,ds = \iint_S (\text{curl}\vec{F}) \cdot \hat{k} \,dA.
\]

Harmonic functions and Maxwell’s equations.

• The Laplacian is the operator $\nabla^2 := \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. It may be applied to functions or vector fields. Notice that $\nabla^2 f = \text{div}(\nabla f)$.
• We say that $f$ is harmonic if $\nabla^2 f = 0$, and similarly for vector fields.
• Let $\vec{E}(x, y, z; t)$, $\vec{H}(x, y, z; t)$ denote the electric and magnetic fields (vector fields in space that change in time $t$). The simplest presentation of Maxwell’s equations (in a vacuum) is:
\[
\nabla \cdot \vec{E} = 0 = \nabla \cdot \vec{H}
\]
\[
\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{H}}{\partial t}, \quad \nabla \times \vec{H} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t}
\]
where $c$ is the speed of light. (The units are not natural in this form and I won’t address them here.) In class, I will say how to derive the wave equations

$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}, \quad \nabla^2 \vec{H} = \frac{1}{c^2} \frac{\partial^2 \vec{H}}{\partial t^2}$$

from them. Notice that this says that, for example, $\vec{E}$ is static (doesn’t change with time) if and only if it is harmonic.