Problem set 10

1. Let \( R = \{ x + iy \mid -1 \leq x \leq 1 - \epsilon, 0 \leq y \leq 1 \} \subset \mathbb{C} \) with \( \epsilon > 0 \) small. Compute \( \int_{\partial R} \frac{dz}{z^5 - 1} \).

2. Given \( f \in Hol(\bar{\mathbb{H}}) \) (closure of upper half-plane), assume there exist \( B, c \in \mathbb{R}_+ \) such that \( |f(z)| \leq \frac{B}{|z|^c} \) (for \( z \in \mathbb{H} \)). Prove that for any \( z \in \mathbb{H} \), \( f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t-z} \, dt \). [Hint: draw a big semicircle in \( \mathbb{H} \).]

3. Determine the poles and residues of \( f(z) = \frac{1}{\sin(z)} \). Find \( \int_{\partial D} \frac{dz}{\sin(z)} \).

4. Let \( P_n(z) = \sum_{k=0}^{n} \frac{z^k}{k!} \). Given \( R \), prove that \( P_n \) has no zeroes in the disk of radius \( R \) for all \( n \) sufficiently large.

5. How many zeroes does \( f(z) = z^{87} + 36z^{57} + 71z^4 + z^3 - z + 1 \) have in the annulus \( 1 < |z| < 2 \)?

6. How many roots of the equation \( z^4 + 8z^3 + 3z^2 + 8z + 3 = 0 \) lie in the right half-plane? [Hint: sketch the image of the imaginary axis and apply the argument principle to a large half-disk.]

7. (a) Suppose \( f \) is a holomorphic function on the unit disk \( D_1 \) which is represented by a power series \( \sum_{n=0}^{\infty} a_n z^n \) there. Also assume that, as a map from \( D_1 \) to \( f(D_1) \), it is injective (hence an analytic isomorphism). Prove that the area of \( f(D_1) \) is given by \( \pi \sum_{n=0}^{\infty} n |a_n|^2 \).

(b) Assume further that \( f \in Hol(D_1) \) (i.e. \( f \) extends to a holomorphic function on some open set containing \( D_1 \)), that \( f(0) = 0 \), and that \( |f(z)| \geq 1 \) if \( |z| = 1 \). Prove that \( \text{area}(f(D_1)) \geq \pi \).