Problem Set 13

For problems (1)-(5), let $f(z) = \sum c_n z^n \in Hol(\mathbb{C})$ and define

$$\alpha_f := \liminf_{n \to \infty} \frac{-\log |c_n|}{n \log n}.$$ 

1. Show that if $f$ has finite order then $\alpha_f > 0$. [Hint: if the order of $f$ is $\lambda_f$ and $\beta > \lambda_f$, show that $|c_n| \leq r^{-n} \exp(r^\beta)$ for $r \gg 0$, and find the minimum value of this expression.]

2. Suppose that $\alpha_f \in (0, \infty)$ and show that for any $\epsilon \in (0, \alpha_f)$ there is an integer $p$ such that $|c_n|^{1/n} < n^{-(\alpha_f-\epsilon)}$ for $n > p$. Conclude that for $|z| = r > 1$ there is a constant $A$ such that $|f(z)| < Ar^p + \sum_{n \geq 1} \left( \frac{r}{n^{\alpha_f-\epsilon}} \right)^n$.

3. Let $p$ be as in (2) and let $N$ be the largest integer $\leq (2r)^{1/(\alpha_f-\epsilon)}$. Take $r$ sufficiently large so that $N > p$ and show that $\sum_{n \geq N+1} \left( \frac{r}{n^{\alpha_f-\epsilon}} \right)^n < 1$ and $\sum_{n=p+1}^N \left( \frac{r}{n^{\alpha_f-\epsilon}} \right)^n < B \exp \left( (2r)^{1/(\alpha_f-\epsilon)} \log r \right)$ where $B$ is a constant which does not depend on $r$.

4. Use (2) and (3) to show that if $\alpha_f \in (0, \infty)$ then $f$ has finite order $\lambda_f$ and $\lambda_f \leq \alpha_f^{-1}$.

5. Prove that $f$ is of finite order iff $\alpha_f > 0$, and if $f$ has order $\lambda_f$ then $\lambda_f = \alpha_f^{-1}$.

6. Use (5) to compute the order of (a) $\cosh \sqrt{z}$ and (b) $\sum_{n \geq 1} n^{-an} z^n$ where $a > 0$.

7. Let $f_1$ and $f_2$ be entire functions of finite orders $\lambda_1, \lambda_2$; show that $f = f_1 f_2$ has finite order $\lambda \leq \max \{ \lambda_1, \lambda_2 \}$. 

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