Problem Set 3

Problems 3-8 use are related to material we will cover in Monday’s lecture.

(1) Let \( \{a_n\} \) be a decreasing sequence of positive numbers approaching 0. Prove that the power series \( \sum a_n z^n \) is uniformly convergent on the domain of \( z \) such that \( |z| \leq 1 \) and \( |z - 1| \geq \delta \), where \( \delta > 0 \). [Use summation by parts.]

(2) For any positive integer \( k \), find a nontrivial formal power series satisfying the Bessel equation \( \{T^2 D^2 + TD + (T^2 - k^2)\} f(T) = 0 \), where \( D \) is formal differentiation. (Take the first nonvanishing term to be \( \frac{t^k}{k!^2} \).) What is its radius of convergence? Conclude that \( \eta \) of your formal series is an analytic solution to the corresponding ordinary differential equation.

(3) Determine all values of \( 2^i \) and \( i^i \).

(4) Find the real and imaginary parts of (a) \( z^z \) and (b) \( \cos(x + iy) \) [use addition formulas].

(5) Describe the Riemann surface associated to \( w = \frac{1}{2}(z + \frac{1}{z}) \) (i.e. the existence domain of \( z(w) \)).

(6) Find open sets where (a) \( \sqrt{1 + z + \sqrt{1 - z}} \) and (b) \( \log(\log(\sqrt{z})) \) are analytic.

(7) Express arctan in terms of log (refer to the treatment of \( \arccos \) in Lecture 9) and write down a maximal set where it is analytic.

(8) Let \( f(z) = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots \). Show that \( f'(z) = \frac{1}{z^2 + 1} \), and conclude that \( f \) represents arctan on the unit disk.