Problem Set 8

(1) Find the coefficients of degree \( \leq 7 \) in the power series expansion of \( \tan(z) \) about \( z = 0 \), by finding the composition inverse of the series for \( \arctan \). [cf. Lectures 20-21].

(2) Prove that the arc of the smallest Poincaré length joining two given points \( a, b \) in the unit disk \( \Delta \) is a circular arc which is perpendicular to the unit circle. [Hint: use a linear transformation carrying one endpoint to the origin, the other to a point on the positive real axis.] Show that this “geodesic distance” is given by \( d_\rho(a,b) = \frac{1}{2} \log \left( \frac{1+|a-b|}{1-|a-b|} \right) \).

(3) Let \( z_0 \in U \) (open set), and \( f \in Hol(U) \) with \( f'(z_0) \neq 0 \). Show that \( \frac{2\pi i}{f'(z_0)} = \int_C \frac{dz}{f(z)-f(z_0)} \) where \( C \) is a small circle centered at \( z_0 \).

(4) Weierstrass’s theorem for a real interval \([a,b]\) states that a continuous function can be uniformly approximated by polynomials. Is this conclusion still true for the closed unit disc, i.e. can every continuous function on the disk be uniformly approximated by polynomials?

(5) Let \( \{a_n\} \) be a sequence of complex numbers. Show that the series \( \sum \frac{a_n}{n^s} \), if it converges absolutely for some complex \( s \), converges absolutely in a right-half plane \( Re(s) > \sigma_0 \), and uniformly in \( Re(s) > \sigma_0 + \epsilon \) for every \( \epsilon > 0 \). Show that the series defines an analytic function in this half-plane. Represent its derivative in series form.