Lecture 22: The Poincaré metric

In this (short) lecture, we give another (more intuitive) view of Schwarz–Pick by introducing a bit of differential geometry.

Let \( U \subseteq \mathbb{C} \) be a region. A conformal metric on \( U \) is a non-negative function \( \rho \in C^\infty(\mathbb{C}) \), or more precisely the expression \( \rho(\bar{z}) \, |dz| \).

For \( z \in U \), \( \beta \in \mathbb{C} \) (think of this as a vector), the length of \( \beta \) at \( z \) (with respect to \( \rho \)) is

\[
\| \beta \|_{\rho, z} := \rho(z) |\beta|.
\]

If \( Y : [a, b] \to U \) is a \( C^1 \) path, the length of \( Y \) (with resp. to \( \rho \)) is

\[
L_{\rho}(Y) := \int_a^b \| Y'(t) \|_{\rho, Y(t)} \, dt.
\]
Example

Let $U = \Delta$. The Poincaré metric is defined by

$$p(z) = \frac{1}{1 - \lvert z \rvert^2}.$$

- $(\Delta, p)$ is a complete metric space (with the distance between two points given by geodesic distance $L_p$ of the "shortest" path) called the hyperbolic disk.

- Geodesics are (generalized) circles intersecting $\lvert z \rvert = 1$ at right angles.

- Historically the first example of a non-Euclidean geometry (Gauss, Bolza, Lobachevsky).
Given two regions equipped with metrics \((U, \rho), (V, \Psi)\),
a holomorphic map \(f: U \to V\) is isometric

\[
\frac{\|\Psi \circ f \cdot \left| \frac{df}{dz}(z) \right| \|_{dz}}{\rho(z)} = 1.
\]

**Theorem** Let \(f: \Delta \to \Delta\) be holomorphic, and \(\rho = \text{Poincaré metric on } \Delta\). Then:

(i) \(f\) is distance decreasing (with respect to \(\rho\)), i.e.

\[
(\rho \circ f) \cdot |f'| \leq \rho
\]

and

(ii) \(f\) is an isometry (i.e. \((\rho \circ f) \cdot |f'\| = \rho\) everywhere) \(\iff f \in \text{Aut}(\Delta)\).
Proof: (i) We want to show
\[
\rho \circ f \quad \left( \frac{\vert f'(z) \vert}{1 - \vert f(z) \vert^2} \right) \leq \frac{1}{1 - \vert z \vert^2}.
\]
But Schwarz–Pick is
\[
\vert f'(z) \vert \leq \frac{1 - \vert f(z) \vert^2}{1 - \vert z \vert^2}.
\]

(ii) \( \implies \): "\( = \)" means equality in Schwarz–Pick, done.

\( \Leftarrow \): Since every holomorphic automorphism of
\( \mathbb{D} \) can be written as a composition of
\( \mu_{e^{i\theta}} \)’s and \( \phi_{a} \)’s, it suffices to check
that:

- \( \mu_{e^{i\theta}} \) is an isometry:
  \[
  \rho \circ \mu_{e^{i\theta}} = \rho, \quad \text{and} \quad \vert \mu_{e^{i\theta}}(z) \vert = \vert e^{i\theta} \vert = 1.
  \]

- \( \phi_{a} \) is an isometry:
  \[
  (\rho \circ \phi_{a})(z) \cdot \vert \phi_{a}'(z) \vert = \frac{1}{1 - \frac{a - z}{1 - \bar{z}a}} \cdot \frac{1 - \vert z \vert^2}{1 - \frac{a - z}{1 - \bar{z}a}^2}.
  \]
\[
\frac{1 - |z|^2}{(1 - \bar{z}z)(1 - \bar{z} - z)(\bar{z} - z)} = \frac{|1 - |z|^2|}{(1 - |z|^2)(1 - |z|^2)} = \frac{1}{1 - |z|^2} = \rho(z).
\]

If you like differential geometry, you’re probably already sold on this. But if not, why should we care about this reformulation, beyond the fact that it simplifies the expression of Schwarz–Pick and makes it more conceptual?

**Corollary** If the closure \( \overline{f(A)} \) belongs to \( \Delta \), then \( f \) has a unique fixed point \( P \), and this \( P \) is the intersection of the sets \( V_i = (f \circ \cdots \circ f)(K) \) for any compact \( K \subset \Delta \).
Proof (Sketch): Let \( \varepsilon = \frac{1}{2} \delta (f(\Delta), \Delta^c) \),
and fix \( z_0 \in \Delta \); then
\[
q(z) := f(z) + \varepsilon (f(z) - f(z_0)) \quad \text{maps} \quad \Delta \rightarrow \Delta
\]
\[\Rightarrow (p \circ q)(z) \cdot |q'(z)| \leq p(z)\]
\[\Rightarrow (p \circ q)(z_0) \cdot |q'(z_0)| \leq p(z_0)\]
\[\Rightarrow (p \circ f)(z_0) \cdot |f'(z_0)| \leq \frac{1}{1+\varepsilon} p(z_0)\]
(for any \( z_0 \in \Delta \), since \( z_0 \) was arbitrary).

So integrating this over geodesics connecting points
\( \alpha, \beta \in \Delta \) yields
\[
d_p(f(\alpha), f(\beta)) \leq \frac{1}{1+\varepsilon} d_p(\alpha, \beta)
\]
\[\Rightarrow f \text{ is a contraction mapping of the complete metric space } (\Delta, p),
\]
at which point the result follows from the contraction mapping fixed point theorem.