1. \( f(z) := \arctan(z) = \sum_{n \geq 1} a_n z^n \), where \( a_n = 0 \) for \( n \) even and \( a_n = \frac{1}{n} (-1)^{(n-1)/2} \). Let 
\( g(z) := \sum_{k \geq 1} b_k z^k \) be the composition inverse of \( f(z) \), i.e., \( f(g(z)) = z = g(f(z)) \).
Then
\[
z = \sum_{n \geq 1} a_n g(z)^n = \sum_{n \geq 1} a_n \left( \sum_{k \geq 1} b_k z^k \right)^n = \sum_{n \geq 1} c_n z^n,
\]
where
\[
c_n = \sum_{k \geq 1, |j| = n} b_k \prod_{i=1}^k a_{j_i} = \sum_{k \geq 1} \sum_{|j| = n} b_k \prod_{i=1}^k a_{j_i}
\]
and \(|j| = j_1 + j_2 + \ldots + j_n\). Thus,

1. \( 1 = c_1 = b_1 a_1 = b_1 \)
2. \( 0 = c_2 = b_1 a_2 + b_2 a_1^2 = 0 + b_2 = b_2 \)
3. \( 0 = c_3 = b_1 a_3 + b_2 (2a_1 a_2) + b_3 a_1^3 = a_3 + 0 + b_3 = -\frac{1}{3} + b_3 \Rightarrow b_3 = \frac{1}{3} \)
4. \( 0 = c_4 = b_1 a_4 + b_2 (2a_1 a_3 + a_2^2) + b_3 (3a_1^2 a_2) + b_4 a_1^4 = 0 + 0 + 0 + b_4 = b_4 \)
5. \( 0 = c_5 = b_1 a_5 + b_2 (...) + b_3 (3a_1^2 a_3 + 3a_1 a_2^2) + b_4 (...) + b_5 a_1^5 = a_5 + 0 + b_3 (3a_3 + 0) + b_5 \)
\[
= \frac{1}{5} + \frac{-1}{3} + b_5 \Rightarrow b_5 = \frac{2}{15} \]
6. \( 0 = c_6 = b_1 a_6 + b_2 (...) + b_3 \)
7. \( 0 = c_7 = b_1 a_7 + b_2 (...) + b_3 (3a_1^2 a_5 + 6a_1 a_2 a_4 + 3a_1 a_3^2) + b_4 (...) + b_5 (5a_1^4 a_3 + 10a_1^2 a_2^2) + b_6 (...) + b_7 a_1^7 \)
\[
= a_7 + 0 + b_3 (3a_9 + 0 + 3a_5^2) + 0 + b_5 (5a_3 + 0) + 0 + b_7 \]
\[
= -\frac{1}{7} - \frac{1}{5} + \frac{1}{9} + \left( \frac{2}{15} \right) \left( \frac{-5}{3} \right) b_7 \Rightarrow b_7 = \frac{17}{315}.
\]

The coefficients of degree \( \leq 7 \) in the power series expansion of \( \tan(z) \) about \( z = 0 \), then, are \( 0, 1, 0, \frac{1}{3}, 0, \frac{2}{15}, 0, \) and \( \frac{17}{315} \). \( \Box \)
Let $L$ be the line (segment) $(-1,1)$ in the unit disk.

It's fairly clear that this is a geodesic with the Poincaré metric $g(z) = \frac{1}{1-|z|^2}$. We also know that the image of $L$ under an isometry is also a geodesic, and that holomorphic automorphisms of $D_1$ are isometries.

So consider $\phi_{ia}(L)$, where $\phi_{ia}(z) = \frac{z - ia}{1 + ia\bar{z}}$, $a \in (-1,1)$.

For $x \in (-1,1)$ we get $\phi_{ia}(x) = \frac{x(1-a^2)}{1+a^2x^2} + i\frac{-a(1+ix^2)}{1+a^2x^2}$.

One checks that this is the portion of the circle with center at $\frac{i1+a^2}{2a}$ and radius $\frac{(1-a^2)}{2a}$ which lies in $D_1$. By the conformality of $\phi_{ia}$ and the fact that $L \subset \partial D_1$, at both endpoints, the circle is $L$ to $\partial D_1$, at the 2 pts. where they meet.

Conversely, if $C$ is any arc of a circle that lies in $D_1$ and is perpendicular to the unit circle at its endpoints, then $C$ arises as the Möbius transformation of $L$. (After a rotation, we can assume that $C$ is symmetric with the y-axis and lies in $D_1$. The endpts. of this arc will certainly have the form $\left(\frac{-1+a^2}{1+a^2}, \frac{-ia}{1+a^2}\right)$, and so we never $\phi_{ia}(L)$.) Hence $C$ is a geodesic.

To compute the geodesic distance between $a \in L \subset D_1$, we need an isomorphism/geometry of $(D_1, g)$ which sends $a \in L$ to points on $L$: $\phi(x) = e^{i\pi \frac{x-a}{1-x^2}}$ does the job, with $\phi(a) = 0$.

Now use $\int_{0}^{x_0} \frac{dx}{1-x^2} = \frac{1}{2} \log \left( \frac{1+x_0}{1-x_0} \right)$, done.
3. Let \( z_0 \in U \), where \( U \) is open, and let \( f \in \text{Hol}(U) \) with \( f'(z_0) \neq 0 \). Then \( f \) is analytic on \( U \), so \( f \) has a power series expansion \( f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \) at \( z_0 \). Let \( g(z) = \sum_{n=0}^{\infty} a_{n+1} (z - z_0)^n \). Now suppose that there isn’t a small open disk \( D_\delta \) centered \( z_0 \) such that \( g(z) \neq 0 \) for all \( z \) in \( D_\delta \). Then for all \( n \geq 1 \), there exists a \( z_n \in D_{1/n} \) such that \( g(z_n) = 0 \). Since \( g \) is continuous and \( z_n \to z_0 \), \( g(z_n) \to g(z_0) \) as \( n \to \infty \). Thus, \( g(z_0) = g(z_0) - g(z_n) \to 0 \), so \( g(z_0) = 0 \), which implies \( a_1 = f'(z_0) = 0 \), a contradiction. Thus, there exists a small circle \( C \) centered at \( z_0 \) such that \( g(z) \neq 0 \) for \( z \) in the region \( \Omega \) bounded by \( C \). Since \( g \) is analytic and non-zero in \( \Omega \), \( 1/g \) is analytic in \( \Omega \). By the Cauchy integral formula,

\[
\int_C \frac{dz}{f(z) - f(z_0)} = \int_C \frac{dz}{(z - z_0)g(z)} = \frac{2\pi i}{g(z_0)} = \frac{2\pi i}{f'(z_0)}.
\]

\( \square \)

**Problem 4.** \( \mathcal{H}(\mathbb{D}) \) is closed under uniform convergence on compact subsets, so a uniform limit of polynomials on the closed disk must be holomorphic on \( \mathbb{D} \). Clearly there exists continuous functions on \( \mathbb{D} \) which are not holomorphic anywhere, such as \( z \to \overline{z} \), which means the *Weierstrass Approximation Theorem* does not hold on \( \mathbb{D} \).

\( \square \)

\( \text{(5)} \)

If \( \sum \frac{a_n}{n^s} \) \( \overline{\mathbb{C}} \) for some \( s_0 \), i.e., \( \sum \frac{|a_n|}{n^{s_0}} \) converges when \( s_0' := \text{Re} (s_0) \), then \( M_n := \left| \frac{a_n}{n^{s_0}} \right| \geq \left| \frac{a_n}{n^{s}} \right| \) \( \forall s \in \overline{U}_{s_0} := \left\{ z \in \mathbb{C} \mid \text{Re}(z) \geq s_0' \right\} \). This gives uniform convergence to an analytic function on \( \overline{U}_{s_0} \) with (homogeneous) derivative \( -\sum \frac{a_n \log n}{n^s} \).

Now you can take the gerb of \( \overline{\mathcal{D}} \) satisfying (5), with two \( \overline{\mathcal{D}} \), and we get the same result on \( \overline{U} \cap \overline{U}_{\overline{\mathcal{D}}} = \overline{U}_{s_0} \).