Problem Set 10

Some of problems 1-5 use material (on localization and prime ideals) we'll cover on Friday.

(1) Prove that an Artinian commutative domain is a field. [Hint: to find an inverse for \( a \neq 0 \), consider \((a) \supset (a^2) \supset (a^3) \supset \cdots \).]

(2) Show that every homomorphic image of a left Noetherian [resp. Artinian] ring is left Noetherian [resp. Artinian].

(3) If \( S \) is a multiplicative subset of a commutative ring \( R \), show that (a) \( S^{-1}(\text{Rad}I) = \text{Rad}(S^{-1}I) \) and (b) \( S^{-1}R \) is Noetherian if \( R \) is Noetherian.

(4) Show that a commutative ring is local if and only if for all \( r, s \in R \), \( r + s = 1 \) implies \( r \) or \( s \) is a unit.

(5) Let \( p \) be a prime in \( \mathbb{Z} \); then \((p)\) is a prime ideal. What can be said about the relationship between \( \mathbb{Z}_p \) and the localization \( \mathbb{Z}_{(p)} \)? Describe \( \mathbb{Z}_{(p)} \) as a subset of the rational numbers.

(6) For \( G = \mathfrak{S}_3 \) (symmetric group), show that \( \mathbb{Q}[G] \cong \mathbb{Q} \times \mathbb{Q} \times M_2(\mathbb{Q}) \) and compute the central idempotents of \( \mathbb{Q}[G] \) which give this decomposition of \( \mathbb{Q}[G] \) into its simple components. Compute, similarly, the decompositions of \( \mathbb{Q}[G_1], \mathbb{Q}[G_2] \) where \( G_1 \) is the Klein 4-group and \( G_2 \) the quaternion group.

(7) Show that, over \( \mathbb{Q} \), \( \mathfrak{A}_5 \) (alternating group) has four irreducible representations ("irreps"), of dimensions 1, 4, 5, 6 respectively.

(8) Show that the 3-dimensional irrep \( \mathbf{st} \otimes \mathbf{sgn} \) of \( \mathfrak{S}_4 \) is equivalent (i.e. isomorphic as \( \mathbb{C}[\mathfrak{S}_4] \)-modules) to the representation of \( \mathfrak{S}_4 \) as the group of rotational symmetries of the cube (or octahedron). [Hint: compute the character of the latter.]

(9) Suppose the character table of a finite group has the following two rows \((\zeta_3 = e^{2\pi i/3})\):

\[
\begin{array}{ccccccc}
1 & 1 & 1 & \zeta_3^2 & \zeta_3 & \zeta_3^2 & \zeta_3 \\
2 & -2 & 0 & -1 & -1 & 1 & 1 \\
\end{array}
\]

corresponding to characters of two irreps. (The first column gives the value on \( \{1\} \), and the remaining columns the values on the other six conjugacy classes.) Determine the rest of the character table.