Problem Set 12

Some of the problems use material we’ll cover on Monday or Wednesday. $R$ is a commutative ring (with 1, as always in this course). Below $K$ denotes an algebraic closure of $k$.

(1) (a) Prove that $\mathbb{C}[z]$ is integrally closed (in its fraction field $\mathbb{C}(z)$), i.e. “normal”. [Hint: suppose you had a $\frac{P}{Q} \in \mathbb{C}(z)$, $P$ and $Q$ relatively prime in $\mathbb{C}[z]$, integral over $\mathbb{C}[z]$] (b) Prove the same result for any UFD.

(2) In class I described “normalizing” the curve $y^2 = x^3 - x^2$ with a nodal singularity at $(0,0)$ by introducing the function “$z = y/x$”. If $R = \mathbb{C}[x,y]/(y^2-x^3+x^2-x+1,z^2-x+1)$, then let $S = \mathbb{C}[x,y,z]/(y^2-x^3+x^2-x+1,z^2-x+1,z^3+y-z)$ with the natural map $\phi : R \to S$. Show (a) that $S \cong \mathbb{C}[z]$ (the coordinate ring of a complex line!). (b) What geometric map does $\phi$ correspond to “pulling back functions” along? Use this to argue that $\phi$ is injective (or prove this by some other means). (c) Show (e.g. using Chinese remainder) that if you use $S' = \mathbb{C}[x,y]/(y^2-x^3+x^2-x+1)$ as I suggested in class, you don’t get a domain, so this can’t be inside $R$’s field of fractions. (d) Use problem (1) to show that $S$ is the integral closure of $R$ in its fraction field. [Hint: don’t make this problem hard. It’s all very simple calculations or trivial arguments.]

(3) Let $R$ be a Noetherian local ring with maximal ideal $m$. If the ideal $m/m^2$ in $R/m^2$ is generated by $\{a_1 + m^2, \ldots, a_n + m^2\}$, show that the ideal $m$ is generated in $R$ by $\{a_1, \ldots, a_n\}$. [Hint: use the result I called the “classical” Krull’s intersection theorem.]

(4) Let $S$ be an integral extension ring of $R$ and suppose $R$ and $S$ are domains. Show that $S$ is a field if and only if $R$ is a field.

(5) Show that every affine $k$-variety in $K^n$ is of the form $V(S)$ where $S$ is a finite subset of $k[x_1, \ldots, x_n]$. [Hint: Use the 1-to-1 inclusion-reversing correspondence between radical ideals and varieties (will do in class), and the Hilbert basis theorem.] (6) If $V_1 \supset V_2 \supset \cdots$ is a descending chain of $k$-varieties in $K^n$, then $V_m = V_{m-1} = \cdots$ for some $m$. [Hint: same as for problem (5).]

(7) If $I_1, \ldots, I_m$ are ideals of $k[x_1, \ldots, x_n]$, then $V(I_1 \cap \cdots \cap I_m) = V(I_1) \cup \cdots \cup V(I_m)$ and $V(I_1 + \cdots + I_m) = V(I_1) \cap \cdots \cap V(I_m)$. [Hint: it may help to use properties of $\text{Rad}(\cdot)$]
(8) A $k$-variety $V$ in $K^n$ is *irreducible* provided that whenever $V = W_1 \cup W_2$ with each $W_i$ a $k$-variety in $K^n$, either $V = W_1$ or $V = W_2$.

(a) Prove that $V$ is irreducible if and only if $J(V)$ is a prime ideal in $k[x_1, \ldots, x_n]$.

(b) Let $K = \mathbb{C}$ and $S = \{x_1^2 - 2x_2^2\}$. Show that $V(S)$ is irreducible as a $\mathbb{Q}$-variety but not as an $\mathbb{R}$-variety.