Problem Set 2

[J] = Jacobson

(1) Show that an algebraically closed field must be infinite.

(2) Suppose that \( K(\alpha)/K \) is a simple extension and that \( \alpha \) is transcendental over \( K \). Show that \( K(\alpha) \) is not algebraically closed.

(3) [J] p. 234 #2

(4) [J] p. 234 #4

(5) Let \( p \) be a prime number. By factorizing \( x^{p-1} - 1 \) over \( \mathbb{Z}_p \), prove Wilson's theorem: i.e., show that \( (p - 1)! \equiv -1 \mod p \).

(6) Let \( K \) be a field of positive characteristic.
   (i) Show that \( K \) is perfect if and only if the Frobenius homomorphism is an automorphism.
   (ii) If \( L/K \) is a totally inseparable extension (i.e. every element of \( L \setminus K \) is inseparable), show that the minimal polynomial of any element of \( L \) over \( K \) is of the form \( x^{p^n} - \alpha \), where \( \alpha \in K \).

(7) Suppose that \( L/K \) is algebraic. Show that there is a greatest intermediate field \( M(\subseteq L) \) such that \( M/K \) is normal.

(8) Suppose that \( L/K \) is finite, with normal closure \( L^c/L \). Show that \( L/K \) is separable if and only if there are exactly \( [L : K] \) embeddings of \( L \) into \( L^c \) fixing \( K \).