Problem Set 7

[J] = Jacobson. (You need to do these over break, but you don’t need to write them up and turn them in.) Problems 6-9 go together, and prove a special case of the Kronecker-Weber theorem (the statement that any abelian extension of \( \mathbb{Q} \), i.e. Galois extension with abelian automorphism group, is a subfield of a cyclotomic field).

1. [J] p. 300 #1
2. [J] p. 305 #3
3. [J] p. 305 #4
4. [J] p. 305 #8
5. The polynomial \( f = x^7 + 9x^2 + 7 \) is irreducible over \( \mathbb{Q} \). By reducing mod 7, show that its Galois group contains a 4-cycle. Use this together with our theorem that any soluble Galois group would have to be contained in \( W_7 \), to show that \( f \) is not solvable by radicals.
6. Let \( p \) be an odd prime. Show that if \( r \in \mathbb{Z} \) then \( \sum_{0 \leq s \leq p} \zeta_p^{rs} \) equals \( p \) if \( r \equiv 0 \pmod{p} \) and equals 0 otherwise.
7. Let \( \tau \) be the Gauss sum \( \sum_{0 \leq n < p} \zeta_p^n \). Show that \( \tau \bar{\tau} = p \). Show also that \( \tau \) is real if \(-1\) is a square mod \( p \), and otherwise purely imaginary.
8. Let \( F = \mathbb{Q}(\zeta_p) \). Show that \( F \) has a unique subfield \( K \) which is quadratic over \( \mathbb{Q} \), and that \( K = \mathbb{Q}(\sqrt{\varepsilon p}) \) where \( \varepsilon = (-1)^{(p-1)/2} \).
9. Show that \( \mathbb{Q}(\zeta_M) \subset \mathbb{Q}(\zeta_N) \) if \( M \mid N \). Deduce that if \( 0 \neq m \in \mathbb{Z} \) then \( \mathbb{Q}(\sqrt{m}) \) is a subfield of \( \mathbb{Q}(\zeta_{4|m}) \).