Problem Set #11

1. Solution:
   Suppose $n = p_1 \cdots p_n$.

   If $T \in \text{Rad}(R)$, which means $\exists x \in T$ s.t. $Tx = 0$ in $R$.

   Then $k^x \in T \implies p_i^{x-\nu(p_i)} \in T \implies p_i^{\max(x, -\nu(p_i))} \in T$.

   Therefore, $\text{Rad}(R) = \bigcap p_i$. 

2. Proof:
   Suppose $I$ is a primary proper ideal. We know that $I$ is included in a maximal ideal, which is of the form $(x)$ for some $x$ in $R$.

   Since $(x)$ is proper, we know $1 \in I$, and $1 \in e \notin I$. We want to show $1 = (x)$.

   Let $D = c_1 \cdots c_n$ be the generatator of $I$, according to $I$ being primary.

   By Cohen's theorem, $R$ is Noetherian.

3. Proof:
   For $f \in \text{Rad}(c_1, c_2)$, we have $c_1 \cdot x \cdot y = c_2 \cdot c_1 \cdot (c_2 \cdot x) = x \cdot c_2 \cdot c_1 = y = c_1 \cdot y$.

   Thus, $\text{Rad}(c_1, c_2) = (c_1, c_2)$. 

4. Solution:
   $I = (x, a, b)$ generates $R$.

   

5. Proof:
   (a) For $I \subseteq P$, let $P = \text{Rad}(P^a)$ be prime. $I \subseteq P^a$ is a partial order set.

   Consider $P_n \leq P_m$. Easy to see $a \cdot b$ is ideal and $I \subseteq P_n \leq P_m$. 

   Suppose $x \in P_m$ and $x \notin P_n$. Then $x \in P_m$ and $x \in P_n$.

   By $P_m$ prime, $x \cdot y \in P_m$ for $k \in j, r \in P_k$.

   For $k \cdot j$, since $r \cdot j \in P_m$, $r \cdot j \in P_n$ by $P_m$ prime, $r \in P_n$.

   Thus, $r \in P_m$, which means $P_m$ is prime.

   By Zorn's lemma, $P$ has a maximal element.

   Thus, there is a minimal prime ideal of $I$. 

   (b) Every proper ideal is contained in a maximal ideal and it is prime.

   By (a), it has at least one minimal prime ideal.

   (c) By $\text{Rad}(I) = \bigcap P$ and for each $I \subseteq P$, we have a minimal prime ideal of $I$.

   This means we can replace each $P$ within all the minimal prime ideals of $I$ included in $P$.

   Thus, $\text{Rad}(I) = \bigcap P$. 

6. Proof. Suppose \( x \in \mathbb{N} \) and \( x \neq N \). By \( \mathbb{N} \) being prime, we have
\[
\exists n.s.t. \, r^n \cdot M \cap N = r^n \cdot \text{ann}(M/N), \quad r \in \text{Rad}(\text{ann}(M/N)) = P.
\]
Thus, we have either \( r \neq 1 \) or \( x = 0 \).
\( \square \)

7. Proof. (a) \( \forall \, x \in \text{ann}(x_0), \, r \in \mathbb{R}, \quad (r \cdot x_0) \cdot x_0 = x_0 = 0 \), \( r \in \text{ann}(x_0) \)

Thus, \( \mathbb{R} \) is a field.

(b) Let \( P \) be a maximal ideal of \( \text{ann}(x_0) \times \sigma(x_0) \)

and say \( P = \text{ann}(x_0) \).

Suppose \( r \in P \) and \( r \neq 0 \).

By \( r \cdot x_0 = 0 \), \( x_0 \in \text{ann}(x_0) \).

According to \( \text{ann}(x_0) \cap \text{ann}(x_0) = \text{ann}(x_0) \cap r \cdot \text{ann}(x_0) \), \( \text{ann}(x_0) = r \cdot \text{ann}(x_0) \).

Let \( r = \text{ann}(x_0) \).

Thus, \( P \) is prime.
\( \square \)

8. Proof. (a) \( \text{ann}(x_0) \) is a partially order set and convex to

\( R \) is a field, hence \( \forall \, x \in \text{ann}(x_0) \). \( r \in \mathbb{R}, \, x \cdot r \in \text{ann}(x_0) \)

must have \( x \in \text{ann}(x_0) \). \( r \in \mathbb{R} \).

By Zorn's lemma, \( \text{ann}(x_0) \) has a maximal element and

by 7(b), it is prime, \( \text{ann}(x_0) \) must be a prime.

Choose \( \tilde{P}_i \) be an associated prime of \( P \).

Let \( \tilde{P}_i = \text{ann}(x_i) \).

We have \( \text{ann}(x_i) = R/P_i \).

If \( M/P_i \neq 0 \),

Choose \( \tilde{P}_i \) be an associated prime for \( M/P_i \).

Let \( \tilde{P}_i = \text{ann}(x_i) \).

Choose \( x_i \) be a prime of \( x_i \) and \( \text{ann}(x_i) \).

\[
\tilde{P}_i = M/P_i + x_i, \quad M/P_i = P_i, \text{ann}(x_i), \text{ann}(x_i) \cap P_i = R/P_i, \text{ann}(x_i) \cap R/P_i = R/P_i.
\]

If \( M/P_i \) nontrivial, \( \text{ann}(x_i) \) is a field and by \( M \) being finite, \( \text{ann}(x_i) \).

Since \( \text{ann}(x_i) \cap P_i = R/P_i, \, \text{ann}(x_i) = R/P_i \), \( \text{ann}(x_i) = R/P_i \), which is a contradiction.

Thus, we must have some \( r \in \mathbb{R} \), \( M/P_i = r \).

\[
M/P_i = r \cdot x_i \cdot x_i \cdot x_i \cdot x_i = x_i/\tilde{P}_i.
\]
9. Proof: \( \Rightarrow \) \( \subseteq \). Let \( P \) be an associated prime of \( M \). \( P = \text{ann}(x) \).

\( \exists \langle x \rangle \in \mathbb{N} \) s.t. \( P^m x = 0 \). \( P^m \text{ annihilates } x \). By \( P \) prime, \( \Rightarrow P \).

\( \Rightarrow \) \( \subseteq \). \( P \times M \) is prime, so \( P = \text{Rad}(\text{ann}(x)) \).

Since \( P \) is prime and \( R \) Noetherian, \( P \) is finitely generated.

Let \( P = (a_1, \ldots, a_n) \).

Since \( P = \text{Rad}(\text{ann}(x)) \), we have \( \exists n \in \mathbb{N} \) s.t. \( a_1^{n} \cdots a_n^{n} x = 0 \).

Consider \( x \langle a_1^{n}, \ldots, a_n^{n} \rangle \cdot x \).

We have \( P = \text{ann}(x) \), which means \( P \) is an associated prime of \( M \). \( \Rightarrow P \).

\( \Rightarrow P \).

\( \subseteq \). \( \exists \langle x \rangle \in \mathbb{N} \) s.t. \( P^m x = 0 \). \( \square \).