Problem Set 8 (Solutions)

(1) Show that if \( B \) is non-degenerate, then for every linear transformation \( T \) of \( V \) into \( V \) there exists a unique linear transformation \( T' \) of \( V \) into \( V \) such that \( B(Tx, y) = B(x, T'y) \) for all \( x, y \in V \). Determine the matrix of \( T' \) in terms of the matrices of \( T \) and \( B \) relative to a base of \( V \). Show that the map \( T \mapsto T' \) is an anti-automorphism in the ring of linear transformations and that \( (T')' = T \) for \( T \) if \( B \) is either symmetric or skew.

Proof. \( B \) is non-degenerate, hence for some basis \( e \) of \( V \), \([B]_e\) is invertible. Then define \( T' \) to be satisfy \([T']_e = [B]_e^{-1} \cdot [T]_e \cdot [B]_e \) which has the desired property. Then we notice such \( T' \) is unique. Suppose not, there is another linear transformation \( T'' \), also has the same property, namely, for any \( x, y \in V \), \( B(Tx, y) = B(x, T'y) = B(x, T''y) \). Choose \( y_0 \in V \) such that \( T'y_0 \neq T''y_0 \). Then for any \( x \in V \), \( B(x, T'y_0 - T''y_0) = 0 \), i.e. \( \text{span}(T'y_0 - T''y_0) \) is the right kernel of \( B \), a contradiction.

For two linear transformations \( S \) and \( T \), \( B(STx, y) = B(TSx, y) = B(x, T'S'y) \), in other words, \((ST)' = T'S'\). To show \( T \mapsto T' \) is surjective, let \( \tilde{T} \) be the linear transform satisfies that \([\tilde{T}]_e = [B]_e^{-1} [T]_e [B]_e \) and we have \( \tilde{T} \mapsto T \). Hence the map is an anti-automorphism.

For any \( x, y \in V \) and let \( B \) be symmetric, \( B(T'x, y) = B(y, T'x) = B(x, T'y) = B(x, T'y) \); if \( B \) is alternating, \( B(T'x, y) = -B(y, T'x) = -B(T'y, x) = B(x, T'y) \). These prove \( (T')' = T \) if \( T \) is symmetric or skew.

(2) Show that

\[ b = \begin{pmatrix} 0 & 2 & -1 & 3 \\ -2 & 0 & 4 & -2 \\ 1 & -4 & 0 & 1 \\ -3 & 2 & -1 & 0 \end{pmatrix} \quad s = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \]

are cogredient in \( M_4(\mathbb{Q}) \) and find a matrix \( p \) such that \( pb \, 'p = s \).

Proof. Note \( b = -'b \) and \( \det(b) \neq 0 \), \( b \) is thus the matrix of a non-degenerate alternating form, and then cogredient to \( s \).

Let

\[ u_1 = [0, -\frac{1}{2}, 0, 0], \]
\[ v_1 = [1, 0, 0, 0], \]
\[ u_2 = [2, 1, 0, 0], \]
\[ v_2 = [-\frac{1}{6}, -\frac{1}{4}, 0, \frac{1}{6}] \]

and the ordered set \( \{u_1, v_1, u_2, v_2\} \) is a symplectic basis of \( \mathbb{Q}^4 \). Then let

\[ M = [u_1, v_1, u_2, v_2] = \begin{pmatrix} 0 & 1 & 2 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix} \]

and \( 'MbM = s \).
Let \((u_1, v_1)\) be a symplectic base for \(V\) and let \(U\) and \(U'\) be the subspaces spanned by the \(u_i\) and the \(v_i\) respectively. Let \(K\) be the subset of \(Sp_n(\mathbb{F})\) of \(\eta\) which stabilize \(U\) and \(U'\). Show that a linear transformation \(\eta \in K\) iff its matrix relative to the base

\[
(u_1, \ldots, u_r, v_1, \ldots, v_r)
\]

has the form

\[
\begin{pmatrix}
A & 0 \\
0 & (A^{-1})^t
\end{pmatrix}, \quad A \in GL_r(\mathbb{F}).
\]

Note that \(K\) is a subgroup of \(Sp_n(\mathbb{F})\).

**Proof.** Let \(e\) denote the basis \(\{u_1, \ldots, u_r, v_1, \ldots, v_r\}\) and if \([\eta]_e\) has such form, it is obvious \(\eta\) stabilize \(U\) and \(U'\) and that \([\eta]_e\) is non-singular. It is left to check \(\eta\) is symplectic. Let \(x, y \in V\) and the matrix of the symplectic form under \(e\) is

\[
[B]_e = \begin{pmatrix}
0 & I_r \\
-I_r & 0
\end{pmatrix}
\]

Then

\[
B(\eta x, \eta y) = 'x \begin{pmatrix}
A & 0 \\
0 & A^{-1}
\end{pmatrix} \begin{pmatrix}
0 & I_r \\
-I_r & 0
\end{pmatrix} \begin{pmatrix}
A & 0 \\
0 & (A^{-1})^t
\end{pmatrix} y
\]

\[
= 'x \begin{pmatrix}
0 & I_r \\
-I_r & 0
\end{pmatrix} y = B(x, y).
\]

Hence \(\eta \in K\).

On the other hand, if \(\eta \in K\) which stabilize \(U\) and \(U'\), it has the form

\[
[\eta]_e = \begin{pmatrix}
M_1 & 0 \\
0 & M_2
\end{pmatrix}, \quad M_1, M_2 \in GL_r(\mathbb{F}).
\]

Moreover, since \(\eta\) is symplectic, we have \('[\eta]_e[B]_e[\eta]_e = [B]_e\), thus we get

\[
'M_1 M_2 = I_r, M_2 = (M_1)^{-1}.
\]

(4) Give an example of a symplectic transformation having no fixed points except for the origin.

Consider \((V, B)\) to be a symplectic space of dimension 2 and \(\{u, v\}\) as the symplectic basis of \(V\). Put \(\eta = \tau_{u_1} \tau_{v_1}\), \(\eta(x) = x + B(x, u)u + B(x, v)v\). \(\eta\) is a composite of symplectic transvections and thus symplectic. We claim \(\eta\) has no fixed points except for the origin. Let \(x \in V\) such that \(x + B(x, u)u + B(x, v)v = x\), then \(B(x, u)u + B(x, v)v = 0\), hence \(B(x, u) = B(x, v) = 0\). \(B\) is non-degenerate and we have \(x = 0\).
Let \((u, v)\) be a symplectic base arranged as in \((1)\) and let

\[
A = \begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix}, \quad A_{ij} \in \mathcal{M}_r(\mathbb{F}).
\]

Show that \(A\) is the matrix of a symplectic transformation iff the \(A_{ij}\) satisfy

\[
\begin{align*}
\dot{A}_{11}A_{22} - \dot{A}_{21}A_{12} &= 1_r = \dot{A}_{22}A_{11} - \dot{A}_{12}A_{21}, \\
\dot{A}_{11}A_{21} - \dot{A}_{21}A_{11} &= 0_r = \dot{A}_{22}A_{12} - \dot{A}_{12}A_{22}.
\end{align*}
\]

**Proof.** Let \(e\) and \(B\) be as in Problem (3). The linear transformation \(A\) represents is symplectic iff

\[
\dot{A}[B]_e = [B]_e,
\]

namely,

\[
\begin{pmatrix}
-\dot{A}_{21} & \dot{A}_{11} \\
-\dot{A}_{22} & \dot{A}_{12}
\end{pmatrix}
\begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix}
= 
\begin{pmatrix}
0 & I_r \\
-I_r & 0
\end{pmatrix}
\]

and we are done. \(\square\)

(6) Finish the proof of the extension-of-basis lemma from the beginning of Monday's class: if \((V, B)\) is a symplectic vector space, then a linearly independent, isotropic subset \(\{u_1, \ldots, u_k\} \subset V\) may be extended to a symplectic basis of \(V\).

**Proof.** Let \((V, B)\) be a symplectic space of dimension \(2r\) over \(\mathbb{F}\) with \(\text{char}\mathbb{F} \neq 2\). Let \(R := U \cap U^\perp\). Then the alternating form on the quotient space \(\overline{U} := U / R\) satisfying \(B(u + R, u' + R) = B(u, u')\) is well-defined since for any \(r, r' \in R\), \(B(u + r, u' + r') = B(u, u') + B(r, u') + B(r, u') + B(r, r') = B(u, u')\). It is also non-degenerate, for that \(R\) is the kernel of \(B|_U\).

Let \(\{\overline{u}_1, \overline{v}_1, \ldots, \overline{u}_k, \overline{v}_k\}\) be a symplectic basis of \(\overline{U}\) and \(\{u_1, v_1, \ldots, u_k, v_k\}\) be any lift of them, which are also linearly independent. Let \(U' = \mathbb{F}(u_1, v_1, \ldots, u_k, v_k)\), and it is clear \(\{u_1, v_1, \ldots, u_k, v_k\}\) is a symplectic basis of \(U'\). We now have \(U = R \oplus U'\) and we put \(\{w_1, \ldots, w_m\}\) to be a basis of \(R\). Hence \(U' \cap R = \{0\}\) gives \(U' \cap (U')^\perp = \{0\}\) in \(V\); together with \(B\) is non-degenerate on \(V\), we know \(V = U' \oplus (U')^\perp\) and that both \(B|_{U'}\) and \(B|_{(U')}^\perp\) are non-degenerate. Therefore it is left to show \(\{w_1, \ldots, w_m\}\) could be extended to a symplectic basis of \((U')^\perp\).

Let \(f_j\) be a functional on \((U')^\perp\) such that \(f_j(w_i) = \delta_{ij}\) and since \(B\) is non-degenerated on \((U')^\perp\), one can find \(s_j \in (U')^\perp\) such that \(f_j(z) = B(z, s_j)\) for all \(z \in (U')^\perp\). Hence we have \(\{w_1, s_1, \ldots, w_m, s_m\}\) such that \(B(w_i, s_j) = \delta_{ij}\) and it is straightforward that \(\{s_j\}\) should be linear independent. Now let \(t_1 = s_1\) and by induction we can make \(t_k = s_k - \sum_{i=1}^{k-1} B(s_i, t_j)w_i\). Then we may check that \(B(t_i, w_j) = 0\) and \(B(t_i, t_j) = \delta_{ij}\). Therefore \(W := \mathbb{F}(w_1, t_1, \ldots, w_m, t_m)\) is an orthogonal sum of hyperbolic planes and \(B|_W\) is non-degenerate, thus \((U')^\perp = W \oplus W^\perp\). One may choose a symplectic basis of \(W^\perp\) and in together we extend \(\{w_i\}\) to a symplectic basis of \((U')^\perp\). \(\square\)
Find the diagonal matrix \( d \) congruent to \( M_3(\mathbb{Q}) \) to

\[
\begin{pmatrix}
-2 & 3 & 5 \\
3 & 1 & -1 \\
5 & -1 & 4
\end{pmatrix}
\]

Also determine a matrix \( p \) such that \( ps'p = d \).

**Solution.** Set

\[
\begin{align*}
v_1 &= \left[1, 0, 0\right], \\
v_2 &= \left[\frac{3}{2}, 1, 0\right], \\
v_3 &= \left[\frac{8}{11}, -\frac{13}{11}, 1\right],
\end{align*}
\]

then \( \{v_1, v_2, v_3\} \) is an orthogonal basis. Set \( p' = [v_1, v_2, v_3] \) and we have

\[
ps'p = \begin{pmatrix}
-2 & 0 & 0 \\
0 & \frac{3}{2} & 0 \\
0 & 0 & \frac{97}{11}
\end{pmatrix}
\]
a. Let \( B(x, y) = \frac{1}{2}(H(x, y) + H(y, x)) \) (this corrects a typo in the book). This is clearly symmetric and bilinear, and for \( a \in K \) we have

\[
B(ax, ay) = \frac{1}{2}(H(ax, ay) + H(ay, ax)) = \frac{1}{2}N(a)(H(x, y) + H(y, x)) = N(a)B(x, y).
\]

b. We must verify \( H(x, y) \) is Hermitian. Linearity in each component is obvious. Since 1 and \( i \) is a basis of \( K \) and \( 1 \in F \), it suffices to verify linearity and conjugate-linearity on \( F \)-multiples of \( i \).

\[
H(aix, y) = B(aix, y) - b^{-1}iB(ax, iy) = b^{-1}B(ax, iy) + iB(ax, y) = aiH(x, y)
\]

A similar argument works for conjugate linearity.

c. If \( H(x, y) = B(x, y) - b^{-1}iB(x, iy) \), then \( H \to B \) sends this to

\[
\frac{1}{2}(B(x, y) + B(y, x) - b^{-1}i(B(x, iy) + B(y, ix)))
\]

Since \( N(i)B(ix, y) = B(bx, iy) \) implies \( B(ix, y) = -b^{-1}B(bx, iy) \), we see \( B(x, iy) + B(y, ix) \) vanishes and we recover \( B \).

Similarly, if \( B(x, y) = \frac{1}{2}(H(x, y) + H(y, x)) \), then \( B \to H \) sends this to

\[
\frac{1}{2}(H(x, y) + H(y, x) - b^{-1}iH(x, iy) - b^{-1}iH(iy, x)) = \frac{1}{2}(H(x, y) + H(x, y) + H(x, y) - H(y, x) = H(x, y).
\]

So the maps are inverses.