1. Solve: \( \mathbb{Z} \)-modules are abelian groups, and simple \( \mathbb{Z} \)-modules are simple abelian groups, say \( \mathbb{Z}_p \) for primes.

By semisimple modules are the direct sum of simple modules, semisimple \( \mathbb{Z} \)-modules are of form \( \bigoplus_{i \in I} \mathbb{Z}/\mathbb{M}_i \) (\( I \) is an arbitrary index set).

2. Proof: In the case \( R = \text{M}_n(D) \), let \( F \) be the center of \( D \). We know that \( F \) is a field.

Need to show \( F \cdot \text{I}_n \) is the center of \( \text{M}_n(D) \).

\( \forall C \in F \cdot \text{I}_n, \quad A \in \text{M}_n(D), \quad \text{Suppose } C \cdot A = A \cdot C \Rightarrow C \in \mathbb{Z}(\text{M}_n(D)). \)

\( \forall B \in \mathbb{Z}(\text{M}_n(D)), \quad \text{Suppose } B \cdot \text{I}_n \). We have \( B(\sum_{i,j} a_{ij} \cdot \text{I}_n) = \sum_{i,j} a_{ij} \cdot B(\text{I}_n) \).

\( B(i,j) = (0, \ldots, 0, b_{ij}, 0, \ldots, 0) \), \( \text{for } (d, e) = (\ldots, 0) \).

This means \( b_{ij} \) for \( i \neq j \) \( b_{ij} \in \mathbb{Z}(D) \).

\( F \cdot (0, \ldots, 0, 1, \ldots, 0) \).

And by \( B(\sum_{i,j} a_{ij} \cdot \text{I}_n) = \sum_{i,j} a_{ij} \cdot B(\text{I}_n) \).

Thus \( B \in F \cdot \text{I}_n \).

In conclusion, \( \mathbb{Z}(\text{M}_n(D)) = F \cdot \text{I}_n \) is a field.

For \( R \) is a semisimple ring, by Artin–Wedderburn's theorem we have

\[ R \cong \text{M}_{r_1}(D_1) \times \cdots \times \text{M}_{r_k}(D_k) \]

\[ \mathbb{Z}(R) \cong \mathbb{Z}(\text{M}_{r_1}(D_1)) \times \cdots \times \mathbb{Z}(\text{M}_{r_k}(D_k)) \]

\[ \cong \mathbb{Z}(D_1) \times \cdots \times \mathbb{Z}(D_k) \]

is a finite direct product of fields. \( \square \)

3. Proof: According to \( R \) is semisimple, by Artin–Wedderburn's theorem we have

\[ R \cong \text{M}_{r_1}(D_1) \times \cdots \times \text{M}_{r_k}(D_k) \] and there are exactly \( r \) non-isomorphic simple modules over \( R \), called \( V_i \).

By \( M \) is a finitely generated \( R \)-module, we can write \( M \) as

\[ M \cong \bigoplus_{i=1}^r V_i^{m_i} \]

Thus we have \( E = \text{End}_R(M) = \bigoplus_{i=1}^r \text{End}_F(V_i^{m_i}) \cong \bigoplus_{i=1}^r \text{M}_n(\text{End}_F(V_i)) \)

\[ \cong \bigoplus_{i=1}^r \text{M}_n(\text{End}_F(V_i)) \]

That is, \( E \) is semisimple. \( \square \)

4. omitted
J. Proof. 

Let \( n \) be a natural number such that \( n^2 = m^2 \cdot d \) and \( n^2 \geq 2n \).

We have a series

\[ 0 \subseteq R_1 = \mathbb{R} \subseteq \cdots \subseteq R_n \subseteq \mathbb{R} \]

and we can construct a C.L. with length \( 2n \).

Suppose \( R = M_n(C) \), \( \text{det} A = d \). Then we have

\[ n^2 = m^2 \cdot d \text{ and } n^2 \geq 2n \text{, or } d = 1 \text{ and } n^2 \geq 2n \text{.} \]

R \subseteq M_n(C) \). \( \Box \)

6. Proof. Suppose \( V \) is a simple \( k[H] \)-module.

Consider \( V \) as a \( k[C] \)-module and \( W \), \( V = W \oplus \overline{W} \) as \( k[H] \)-module.

Let \( f : V \rightarrow W \), \( f \in \text{Hom}_{k[C]}(V, W) \). \( f \circ (v) = id \).

Define \( \pi(v) = \frac{1}{|C:H|} \sum_{g \in (C:H)} g^{-1} f(g \cdot v) \). \( (\pi(v) \) is well-defined. Since \( g \circ h \cdot v = g \cdot h^{-1} f(g^{-1} h \cdot v) = g \cdot f(h \cdot v) = g \cdot f(h \circ \overline{h}) \).

1. \( \pi \in \text{Hom}_{k[H]}(V, W) \).

Let \( x \in C \). \( \pi(x \cdot v) = \frac{1}{|C:H|} \sum_{g \in (C:H)} g^{-1} f(g \cdot x \cdot v) \)

\[ = \frac{1}{|C:H|} \sum_{g \in (C:H)} g^{-1} f(x \cdot g \cdot v) \]

\[ = x \cdot \pi(v) \).

2. \( \pi(w) = \frac{1}{|C:H|} \sum_{g \in (C:H)} g^{-1} f(g \cdot w) = \frac{1}{|C:H|} \cdot (C:H) \cdot w = w \)

Thus \( V = W \oplus \ker \pi \) which \( \pi \) is semi-simple. \( \Box \).