IV. Properties of MT domains

To conclude these talks I'll discuss several aspects of MT domains, mainly in the context of the $\text{U}(2,1)$ domain described last time:

- CM points
- boundary components (and "nilpotent orbit" VHS)
- automorphic cohomology (only briefly).

A. The ball model

Recall our main working example:

$$V = \text{6-dim 1 Q-vector space}$$

$$F = \mathbb{Q}(\sqrt{-d})$$

$$Q : V \times V \to \mathbb{Q}$$ alternating nondegenerate bilinear form

$$\mu : F \to \text{End}_F(V) \text{ w/ eigenspaces } V_+ \oplus V_- = V_F$$

$$V_{+} = V_{3,0} \oplus V_{1,2} \oplus V_{1,2} \oplus V_{0,3}$$

such that

- $V_{3,0}$ is $V_{+}$-invariant
- $\dim(V_{2,1} \cap V_{+}) = 1$

Such $\mu$ are equivalent to decompositions $V_{+,a} = V_{3,0} \oplus V_{2,1} \oplus V_{1,2}$ satisfying a positivity condition (discussed below).

picture:
$$
\begin{array}{c}
V_+ \\
V_-
\end{array}
\begin{array}{cccc}
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\times & \times & \times & \times \\
\end{array}
$$

The diagram above illustrates the decomposition of $V_{+}$ with respect to the action of $\mu$. The circles and squares represent the eigenspaces $V_{3,0}$, $V_{2,1}$, and $V_{1,2}$, respectively.
Write \( \lambda(v,w) := -i \Omega(v,w) \) \((v,w \in V_+, \epsilon)\).

To simplify things, make the following choice of \( \Omega \):
- \( \{ e, f, v_1, v_2, v_3 \} = V_+ \) be an \( F \)-basis
- assume \( [v_1]_e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), so that

\[
\lambda(v,w) = [v]_e [v_1]_e [\bar{w}]_e.
\]

Set \( M := \mathbb{F}_p(V, \Omega) \cap \text{Res}_{\mathbb{F}_p/\mathbb{Q}}(\text{GL}(V_+)) \), and note

\[
M(\mathbb{R}) \cong \text{U}_\lambda(V_+, \epsilon) \equiv \{ g \in \text{GL}(V_+, \epsilon) \mid g^{-1} \lambda g = \lambda \}.
\]

Let \( \varphi_0 \) be the \( \Omega \)-polynomial HS defined by

\[
\begin{align*}
\varphi_0 &= \{ w_0 = y_2, \\
\omega_1^+ = y_1 + y_3, \\
\omega_1^- = y_1 - y_3 \}.
\end{align*}
\]

We may identify \( \omega_1 \) with \( \omega_1 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \) under \( \lambda \), so that under \( g \in M(\mathbb{R}) \)

\[
g(p, l) = (g \cdot p, g \cdot l)
\]

is \((g \omega_0, g \omega_1^-)\).

In order to bypass computing \( \lambda(\cdot, \cdot) \), it's nice to have a formulation of \( l \) as a row vector, so that we can just dot it against \( p \) viewed as a column vector.

\[
\lambda(g \cdot p, g \cdot l) = \lambda(\cdot, \cdot)
\]

\[
[g]_y \cdot \overline{[g]_y} = \overline{[g]_y} \cdot [g]_y
\]

\[
\overline{[g]_y} \cdot [g]_y = [g]_y \cdot \overline{[g]_y}
\]
Since \( \mathcal{L} \) is defined by 
\[
\iota \left[ \omega_1 \right]_y \left[ h \right]_y \left[ \eta \right]_y = 0,
\]
the new vector is 
\[
\iota \left[ \omega_1^{-1} \right]_y \left[ h \right]_y \rightarrow \iota \left( \left[ g \right]_y \left[ \omega_1^{-1} \right]_y \right) \left[ h \right]_y = \iota \left[ \omega_2^{-1} \right]_y \iota \left[ g \right]_y \left[ h \right]_y = \left( \iota \left[ \omega_2^{-1} \right]_y \left[ h \right]_y \right) \left[ g \right]_y.
\]

The first formulation of our "bell model" then goes as follows:
writing 
\[
\tilde{\rho} := \left[ \omega_0 \right]_y, \quad \tilde{\mathcal{L}} := \iota \left[ \omega_1^{-1} \right]_y \left[ h \right]_y, \quad \left\langle \tilde{\omega}, \tilde{\omega} \right\rangle := \iota \left[ h \right]_y,
\]

\[
\begin{pmatrix}
\tilde{u} \\
\tilde{v} \\
\tilde{w}
\end{pmatrix}
\]

\( \tilde{u}, \tilde{v} \in \mathbb{C} \)

\[
\mathcal{D}_M := \left\{ \left( \tilde{\rho}, \tilde{\mathcal{L}} \right) \in \mathbb{C}^3 \times \mathbb{C}^3 \mid \tilde{\omega} \left( \tilde{\rho} \right) = 0, \quad \left\langle \tilde{\omega}, \tilde{\omega} \right\rangle > 0, \quad \left\langle \tilde{\mathcal{L}}, \tilde{\mathcal{L}} \right\rangle > 0 \right\}
\]

\[
\mathbb{C}^* \times \mathbb{C}^*
\]

\[
\mathbb{C}^* \times \mathbb{C}^*
\]

\[
\mathbb{C}^* \times \mathbb{C}^*
\]

\[
\left( \text{with } g \in \mathbb{G}(\mathbb{C}), \text{ this semi-algebra works on } \mathcal{D}_m. \right)
\]

A holomorphic mapping \( \Delta \rightarrow \mathcal{D}_M \)

\[
\text{is a VHS } \text{ if and only if } \iota \left( d\tilde{\rho} / d\tilde{q} \right) = 0.
\]
B. CM points

Definition: (a) A HS \( p \in D \) is \( \text{CM} \) (has complex multiplication)
\( \Leftrightarrow M_p \) is abelian. [Such HS always come from AC,]
and are dense in \( D \).
(b) A CM field \( L \) is a finite extension of \( \mathbb{Q} \) with \( \rho \in \text{Gal}(\mathbb{Q}) \)
s.t. \( \forall \Theta \in \text{Hom}(L, \mathbb{Q}) \) \( \Theta = \Theta \circ \rho \).

The isogeny subgroup of a HS inside its MTC always contains the maximal tori, so for CM HS, \( M_{\rho^p} \) \( \rho = \{ \rho \} \). By what we said about the MTC NC locus \( \xrightarrow{\text{NC domain}} \) \( \text{MTC domain} \) relationship in part III, this means CM HS are isolated in moduli. So: what's the relationship between (a)/(b)?

Theorem: (a) All irreducible CM HS are constructed as follows:
- \( V \rightleftharpoons L \) (CM field) viewed as \( \mathbb{Q} \)-v.s.
- (ii) = partition of \( \text{Hom}(L, \mathbb{Q}) \) into “Hodge 4/1’s” which are used
to assign Hodge type (\( p,q \)) to the eigenspace in
  \[
  V_\Theta = L \otimes \mathbb{Q} \Theta \subseteq \bigoplus_{\Theta \in \text{Hom}(L, \mathbb{Q})} V_{\Theta, \mathbb{Q}}
  \]
  (i.e., \( \bigoplus_{\Theta \in \text{Hom}(L, \mathbb{Q})} V_{\Theta, \mathbb{Q}} = V(p,q) \)).
- (b) They are all polarized via \( \mathbb{Q} (\alpha(l_1), \alpha(l_2)) := \text{Tr}_{L/\mathbb{Q}} (\delta \cdot l_1 \cdot p(l_2)) \)
  where \( \delta \in \text{L}^* \) is an appropriate element.

Idea of proof: Since \( V \) rank., \( E = \text{End}_{\mathbb{Q}} (V) \) is a division algebra/\( \mathbb{Q} \)
with “Rosen form,” \( T \) defined by \( \mathbb{Q} (E_v, w) = \mathbb{Q} (v, E^v w) \).
From the fact that \( M \) is abelian and \( V \) rank., one shows \( E \) is abelian,
have a fixed; one then uses \( T \) to construct \( p \).
In our local model case, all the CM HS will be constructible as follows. Let $L_0$ be a totally real cubic field, $L := L_0(\sqrt{-d})$, and order its complex embeddings $\Theta_1, \Theta_2, \Theta_3, \overline{\Theta}_1, \overline{\Theta}_2, \overline{\Theta}_3$ so that $\mathbb{F} = \mathbb{Q}(\sqrt{-d})$, $\Theta_1, \Theta_2, \Theta_3$ refer to $\mathbb{F}$. Then the partition $\mathcal{P}$ identifies one $\Theta_i$ with $(3, 0)$, one $\Theta_j$ with $(2, 1)$, and one $\Theta_k$ with $(1, 2)$, so that

$$V_C = \left( \bigoplus_{l=1}^{3} V_{\Theta_l, 0} \right) \oplus \left( \bigoplus_{l=1}^{3} V_{\overline{\Theta}_l, 0} \right).$$

The corresponding points of $\mathcal{E}_m$ will consist of vector pairs $\left( \tilde{p}, \tilde{e} \right)$ with coordinates in $L^C$ (complex closure), but not all pairs defined in $L^C$ will be of CM type. (Furthermore, an additional constraint really needs to be applied to the above construction.)
C. Boundary components

Recall that for a PVHS \((M, \Phi, Q) / \Delta^*\) with unipotent monodromy \(T\)

\[
\left( \begin{array}{c}
\text{coord. } q, \\
\tau = \log q = \frac{\log q}{2 \pi i}
\end{array} \right)
\]

we can trivialize the local system by \(e^{tN} M\),

with extends to \(\tilde{\mathcal{X}} / \Delta\). Schmid proved that \(\tilde{\Phi}\) extends to holomorphic subbundles of \(\tilde{\mathcal{K}}_0 = \tilde{\mathcal{X}} \otimes \mathcal{O}_\Delta\), and we write \(\tilde{F}_0^* = F_0^{-1} \mathcal{O}_\Delta\). The LMHS is then \((\tilde{M}_0, \tilde{F}_0^*, M, Q)\), but by the construction of \(\tilde{\mathcal{K}}_0\) it really depends on \(q\). The

souling out the dependence on \(q\) leads to the following

**Definition 2:** The nilpotent orbit associated to \((M, \Phi, Q)\) is \((H, e^{tN} F_0^*, Q)\)

\((= \text{ a PVHS on a possibly smaller disc})\). This is the closest thing
to a "constant" VHS possible when the local system has monodromy; relative
to \(e^{tN} M\), \(e^{tN} F_0^*\) is constant.

**Remark:** Recall that a VHS has \(\nabla_{\Phi} E \subset \mathcal{X}'_\Delta \otimes F_0^{-1} (\forall \Phi) \subset \Delta^*\)

On \(\Delta\), we have \(\nabla_{\Phi} E \subset \mathcal{X}'_\Delta (\log Q) \otimes F_0^{-1} (\forall \Phi)\)

with \(\text{Res}_\delta (\Phi) = \frac{N}{2 \pi i} \).

\(\Rightarrow \quad NF_0^* \subset F_0^{-1}\).

Next we have to discuss the pertinent compactifications of \(K\)-sor"...
Definition 3: A nilpotent cone $\sigma \subseteq M_n$ is a strongly convex, e.g. rational polyhedral cone with commuting nilpotent generators.

Fix a nilpotent cone $\sigma = \Sigma_{j=1}^{n}(R_2 \circ N_j) \subseteq M_n$, and let

$$F^* \in D_M = M(\sigma).F_y \subseteq D$$

be given.

**Definition**: $e^{\sigma} F^* \subseteq D_M$ is a $\sigma$-nilpotent orbit if

1. $e^{\Sigma_{j=1}^{n}(y_j N_j)} F^* \in D_M$ for $y_j \gg 0$ \((\forall j)\) \([\text{polarizability}])$
2. $N F^* \subseteq F^{-1}$ \([\text{horizontality}])$

Next let $\Sigma$ be a fan of nilpotent cones in $M_n$ and define

the set of nilpotent orbits

$$D_{M, \Sigma} = \{ (\sigma, Z) \mid \sigma \in \Sigma; Z \text{ is a } \sigma \text{-nilpotent orbit} \}$$

$$U = \{ (\sigma, \emptyset) \}$$

$$D_{M, \emptyset} = \emptyset$$

This is really a way of parametrizing limits up to rescaling (or rather, $D_{M, \Sigma} \setminus D_M$ is).

Let $D_{M, \sigma} := D_M \{ \text{ for a family of all cones } \sigma \} \subseteq D_M$.

Next we consider a given $P \subseteq M(\Sigma)$.

**Definition 5**: $P$ is compact with $\Sigma \ni \text{Ad}(g)\sigma \in \Sigma \cup \{ P \circ \sigma \}$ and strongly compact with $\Sigma \ni \text{compact w./ } \Sigma$ and if $g \sigma \in \Sigma$

$$\exists k_1, \ldots, k_n \in \mathbb{Z} \ni \text{Exp}(g) \subseteq \bigcap k_i \sigma \subseteq \Sigma.$$ 

The partition compactification is then defined as a set by

$$(\sigma) \backslash D_M, \Sigma$$.
and for the case where $D$ has a topological PVS (Hilbert symmetry), this does recover the (algebraic) compactification of Kob-\-Mumford-\-Repressent-Tai. Otherwise one gets stuck at the boundary 

\[
\frac{c_2(10\times 6)}{1100}
\]

since the infinitesimal period relation on $D$ becomes "non-infinitesimal" at the boundary. (I won't say how to topologize (E), which gets into the "torsors" E.)

Back to the $6\times 11$ model

We want to understand the set of nilpotent orbits — which will give us both a window onto $VTS \subset D$ and into boundary components.  

Intentions: the possible $LHT$ are clear, from

\[
\begin{align*}
N^3 &= 0 \\
N^2 &= 0
\end{align*}
\]

Segreality holds like

\[
\begin{align*}
l & \in \mathbb{P} \\
p & \text{varies}
\end{align*}
\]
We first determine the possible nilpotent cases:

\[ m = \text{Lie} (MCR) \cong \left\{ g \in \text{End}(V_t, C) \left| \tau [g]_y [b]_y + [b]_y \tau [g]_y = 0 \right\} \right. \]

\[ \begin{pmatrix} A & B & C \\ D & E & \overline{B} \\ C & \overline{D} & -\overline{A} \end{pmatrix} \left| \begin{array}{c} C, E, G \in \mathbb{R} \\ A, B, D \in \mathbb{C} \end{array} \right. \]

We state w/o proof the following fact:

**Lemma:** Any nilpotent case can be conjugated (by \( G(\mathbb{R}) \)) to one lying of the form

\[ \begin{pmatrix} 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \]

So

\[ \Sigma \subset \left\{ \begin{pmatrix} 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \left| b \in \mathbb{R} \right. \right\} =: \sigma \quad \text{not yet a nilp. cat.} \]

\[ \overline{N, B} \colon \exp (\sigma_0) = \left\{ \begin{pmatrix} 1 & \alpha & i \beta \\ \overline{\alpha} & 1 & \overline{\beta} \\ \beta & \overline{\beta} & \overline{\alpha} + \beta \end{pmatrix} \left| \begin{array}{c} \alpha, \beta \in \mathbb{R} \end{array} \right. \right\} \]

\[ \sigma \text{ is commutative iff for } N_1, N_2 \in \sigma \quad (N_1 \sigma_0 = (0 \ 0 \ 0 \ 0)) \]

\[ N_1N_2 - N_2N_1 = \left( \begin{array}{ccc} 4 & 2 \bar{z} & -2 \bar{z} \\ -2z & 4 & \bar{z} \\ 2 \bar{z} & \bar{z} & 4 \end{array} \right) = 0 \]

When does \( \Sigma \) have a nilpotent orbit?

Well, we need for there to exist \((\tilde{r}, \tilde{x}) \in \tilde{D}_m \) s.t.

\[ \tilde{L}(N \tilde{r}) = \left( \begin{array}{ccc} x_j & i b_j & x_j \\ -i b_j & -x_j & x_j \\ x_j & x_j & x_j \end{array} \right) \left( \begin{array}{c} \tilde{r} \\ \bar{r} \\ 0 \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right) \]
Claim: $x \neq 0$.

[This is because we need $\{ e^{\mathbb{Z} \cdot \lambda} \}$ to eventually lie $(\text{Int} \gg 0)$ in $B^c$. Because of the form of $[m]_y$, this will never happen if $x = 0$.]

So (projectively) we may write $\hat{p} = \begin{pmatrix} x \\ y \\ i \\ \bar{z}_1 \bar{z}_2 \\ \bar{z}_1 \\ \bar{z}_2 \\ 0 \end{pmatrix}$, which together with

\[
\begin{pmatrix}
\bar{a}_1 \\
\bar{a}_2 \\
0
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
\bar{a}_1 \\
\bar{a}_2 \\
0
\end{pmatrix}
\]

must be killed under the dot product $u_i \\ \bar{z}_i^{}$. Hence these 3 vectors are dependent; in particular, the latter two must be, and so

\[
0 = \bar{a}_1 \bar{a}_2 y + i b_1 \bar{z}_2 - \bar{a}_1 \bar{a}_2 y - i b_2 \bar{a}_1
\]

\[
= (\bar{a}_1 \bar{a}_2 - \bar{a}_1 \bar{a}_2) y + i (b_1 \bar{a}_2 - b_2 \bar{a}_1)
\]

\[
= 0 \quad \text{(from above)}
\]

\[
\Rightarrow b_1 \bar{a}_2 = \frac{\bar{z}_1}{\bar{a}_2} = \frac{\bar{z}_2}{\bar{a}_2} \quad \Rightarrow \quad N_1 \text{ is a multiple of } N_2.
\]

So the only nilpotent cases yielding $\sigma$-nilpotent orbits are 1-dimensional.

2 cases:

- **Type (III):** $\sigma = \mathbb{R}_{\geq 0} \begin{pmatrix}
0 \\
0 \\
\bar{a}_2 \\
\bar{a}_2 \\
0 \\
0 \\
0
\end{pmatrix}$, $a \neq 0$.

  - *Working example:*
    \[
    \begin{pmatrix}
    0 \\
    0 \\
    1 \\
    0 \\
    0 \\
    0 \\
    0
    \end{pmatrix}
    \]

- **Type (II):** $\sigma = \mathbb{R}_{\geq 0} \begin{pmatrix}
0 \\
0 \\
\bar{a}_2 \\
\bar{a}_2 \\
0 \\
0 \\
0
\end{pmatrix}$, $b \neq 0$.

In each case we would like to compute the boundary coset

\[
\Gamma_\sigma^* := M(\Gamma_\sigma) \backslash P / P \cap \text{Stab}(\sigma) \quad U_\sigma^* \backslash \mathbb{R}^n \quad \text{via Ad}
\]
What are these stabilizers?

**Type (III):**

\[
\begin{pmatrix}
1 & \alpha & \beta \\
1 & \frac{1}{\alpha} & \frac{1}{\beta}
\end{pmatrix}
\begin{pmatrix}
1 & -\alpha & -\beta \\
1 & -\frac{1}{\alpha} & -\frac{1}{\beta}
\end{pmatrix}
= \begin{pmatrix}
1 \\
1
\end{pmatrix}
\]

ok

**Type (III):**

So \( \Gamma_c = \left\{ \begin{pmatrix}
1 & \alpha & \beta \\
1 & \frac{1}{\alpha} & \frac{1}{\beta}
\end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\} \), type (II)

\[
\left\{ \begin{pmatrix}
1 & \alpha & \beta \\
1 & \frac{1}{\alpha} & \frac{1}{\beta}
\end{pmatrix} : a^2 = \text{Re}(\beta) \right\}
\]

\( \text{type (III)} \)

To determine the nilpotent orbits (hence the boundary components),

write \( z = r + is \), \( r = 0 \) for simplicity. We want that the positivity

conditions are satisfied for

\( (\hat{\rho}, \hat{\lambda}) = e^{isN} (\tilde{\rho}, \tilde{\lambda}) \) for \( s \gg 0 \),

\( \text{i.e.} \)

\[
\begin{align*}
|y|^2 & < 2 \text{Re} (\chi \bar{z}) \\
|\lambda|^2 & < 2 \text{Re} (\chi \bar{z})
\end{align*}
\]

**Type (III):**

\( N = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \), \( e^{isN} = \begin{pmatrix}
1 & is \\
is & 1
\end{pmatrix}
\]

\[
\Rightarrow \tilde{\rho} = \begin{pmatrix}
x + isy - \frac{s^2 z}{2} \\
y + iz
\end{pmatrix}
\]

\( e^{isN} (\tilde{\rho}) \)

(\#) says

\( y^2 + isz + is\bar{y} + s^2 |z|^2 > 2 \text{Re}(xy \bar{z}) + 2 \text{Re}(\bar{z} y \bar{z}) - 2 \text{Re}(\frac{s^2 |z|^2}{2}) \)

(\( \Leftrightarrow z \neq 0 \))
\( \dot{\gamma} = (u + v + w) e^{-i\Sigma N} = (u, -i(u + v) - \frac{s^2}{2}u - iuv + uv) \)

\((\star \star) \quad \text{say} \quad s^2 |w|^2 - 15uv + 15uv + |v|^2 \geq \Re \left( -\frac{s^2}{2} (w^2 + 15uv + u\overline{v}) \right) \)

\(\iff u \neq 0 \)

So (projectively) we may normalize

\[ \tilde{\rho} = \begin{pmatrix} x_0 \\ y_0 \\ 1 \end{pmatrix}, \]

and by moving in the multiplet orbit we may take

\[ \tilde{\rho} = \begin{pmatrix} x \\ 0 \\ 1 \end{pmatrix}. \]

Since \( \tilde{\xi} \) must (still) \( \tilde{\rho} \) and \( N\tilde{\rho} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \)

\[ \tilde{\lambda} = (1, 0, -X). \]

Applying \( \tilde{\rho} \in \begin{pmatrix} 1 & a & -a \\ 1 & 1 & 1 \end{pmatrix \}

\( \begin{pmatrix} X \\ 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} x + \beta \\ a + \beta \\ 1 \end{pmatrix} \)

we see that

\[ \tilde{\rho} \in \begin{pmatrix} 1 & i & -i \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} X + i\Sigma N \beta \\ 0 \\ 1 \end{pmatrix} \]

we see that

\[ \tilde{\rho} \in \begin{pmatrix} 1 & i & -i \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} X + i\Sigma N \beta \\ 0 \\ 1 \end{pmatrix} \]

\( \tilde{\rho} \in \mathbb{C}^* \)
Type(II) (the interesting one!) Assume $b > 0$ \( (b < 0 \text{ is similar but not the same}) \)

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = \begin{pmatrix} C & B \end{pmatrix} \begin{pmatrix}
0 \\
x \\
z
\end{pmatrix} = \begin{pmatrix} \frac{x}{y} \\
1 \\
z
\end{pmatrix} = \begin{pmatrix} \frac{x - sbz}{y} \\
1 \\
z
\end{pmatrix}
\]

\[
\begin{pmatrix}
u \\
v \\
w
\end{pmatrix} = \begin{pmatrix} u & v & w + sbu \\
0 & 1 & 0
\end{pmatrix}
\]

\[(**): \quad |v|^2 > 2 \text{Re}(x\bar{v}) - 2 \text{Re}(sb|z|^2) \quad \iff \quad z \neq 0, \quad u = 0,
\]

Wolog put $z = 1 = v$, so the orbit takes the form

\[
\begin{pmatrix}
x - sb \\
y \\
z
\end{pmatrix}, \quad \begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = \begin{pmatrix} 0 & 1 & u \\
0 & 1 & 0
\end{pmatrix}
\]

fixed!! - exactly as expected. \((b < 0 \Rightarrow \tilde{p} \text{ fixed})\)

We can "normalize" each nilpotent orbit by taking the point where $x = 0$:

\[
\tilde{p} = \begin{pmatrix} 0 \\
y \\
1 \end{pmatrix}, \quad \tilde{v} = \begin{pmatrix} 0 & 1 & -y \\
0 & 1 & 0 \end{pmatrix}
\]

\[
\sum_{-1 \text{ con.}} \text{ points of } \mathcal{D}_0 / \mathcal{D}
\]

Then quotient by the action of

\[
\begin{pmatrix} 1 & \beta y \\
\beta & y + \beta \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\
y + 2
\end{pmatrix}
\]

\[
\begin{pmatrix} 0 & 1 & -y \\
0 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & -y + \beta \end{pmatrix}
\]

So, as $x \in \mathcal{O}_F$, we get

\[
\mathcal{D}_0 / (\mathcal{D}_0 / \mathcal{D}) \cong \mathbb{C} / \mathcal{O}_F \cong \text{CM elliptic curve}!!
\]
D. Automorphic Cohomology

We said above that

\[ D_M \text{ Hermitian symmetric} \rightarrow \Gamma \]

\[ \text{e.g., if IPR=0} \]

\[ \Gamma \text{ quasi-projective algebraic variety} \]

It is essentially quotient of \(\Gamma\)-invariant sections in

\[ (H^0(D_M, K_{D_M} \otimes 1))^{\Gamma} \]

which yield the function embedding \(\pi^* \mathbb{D}_M \rightarrow \mathbb{P}^N\) (which also yields the Boîly-Boyd compactification). We can even choose the bases of sections in such a way that the image is defined \(\overline{\mathbb{Q}}\) and the functions take \(\overline{\mathbb{Q}}\)-values on CM points \(D_m\).

Ex: the ball domain \(B\) above (not what we called \(D_m\)), which can be constructed as a classical \((3,3)\) space for \(\mathbb{H}\) with Hodge numbers \((3,3)\). The functions are quotients of Picard modular forms.

When the IPR is non-trivial, it is usually the case that

\[ D_M \text{ is not Hermitian symmetric but fibers non-holomorphically over something that is:} \]

\[ D_M = M(R)/H \rightarrow M(R)/K \]

\[ \text{not maximal compact} \rightarrow \text{maximal compact} \]
For example the "ball model" (IV.A) fiber over the ball
\[ D_m = \frac{U(2,1)}{T} \rightarrow \frac{U(2,1)}{K} = B \]
via
\[(\tilde{\rho}, \tilde{\lambda}) \mapsto (\tilde{\rho})^\perp \cdot \tilde{\lambda} \]
which correlates to merging \( V_{g,t}^2 \) and \( V_{g,+}^2 \) \( \sim \) \( V_{g,t}^{2,1} \) \( \sim \) \( V_{g,+}^{2,1} \) yielding a \( \mathbb{Q} \) of type \((2,3)\).

Assume for the rest of the discussion that \( D_m = M(R) / T \)

where \( T \) is a maximal torus, and write \( K \) for a maximal
compact subgroup of \( M(R) \) containing \( T \), put \( q := \frac{1}{2}(\text{dim}_R K - \text{dim}_R T) \).

Let \( L_\lambda \rightarrow D_m \) denote the line bundle associated to a weight \( \lambda \in X^*(T) \).

**Theorem (Schub):** (i) \( H^p(D_m, O(L_\lambda)) = 0 \) if \( p \neq q \.

(ii) For \( p = q \), these spaces realize
discrete series representations of \( M(R) \) —

i.e. the infinite dimensional responsible
representations occurring in the right regular representation of \( M(R) \)
on \( L^2(M(R)) \). [These are fundamental objects in representation
theory, pioneered by Harish-Chandra and of great interest in the Langlands program.]

\( \updownarrow \) for \( \lambda \) regular antidominant.
Now the "automorphic" aspect of pure cohomology groups is not yet in view: automorphic cohomology is

\[ H^q(D_m, \mathbb{Z}) \cong H^q(\mathbb{R}^m, \mathbb{Z}) \]

and more importantly there is the cuspidal automorphic cohomology

\[ \mathcal{S}^q(\Gamma_m, \mathbb{Z}) \cong H^q(\mathbb{R}^m, \mathcal{Z}) \]

which has a decomposition into $K$-eigenvalues of Lie algebra cohomology groups

\[ \bigoplus_{\tau} H^q(\mathbb{R}^m, \mathcal{Z}) \oplus \bigoplus_{\tau} H^q(\mathbb{R}^m, \mathcal{Z}) \]

It occurs in the HC-module - decomposition of the automorphic forms $\mathcal{A}(\mathbb{R}^m, \Gamma)$ under the right regular action of $(\mathbb{R}^m, K)$.

H. Carayol recently discovered that for a particular choice of $\lambda$ in the ball model

\[ \mathcal{S}^1(\mathbb{R}^m, \mathcal{Z}) \]

comes from $\tau$ in the limit of discrete series, which is a part of the representation theory of $M(\mathbb{R})$ where the Langlands program has so far not been able to say much. For a more general class of $\tau$, and more importantly, he has the following result: for which we let $\mathcal{E}$ for $\mathcal{E}$ denote a type(II) Kato-Usui boundary compact of $\mathbb{R}^m$.
Theorem (Carayol): Define Fourier coefficients for classes in $(\mathfrak{c})$ by

$$H^i \left( \mathfrak{c} \right) \rightarrow H^i \left( \mathfrak{c} \mid \mathfrak{c} \right) \rightarrow H^i \left( \mathfrak{c} \right)$$

and say $\omega$ is arithmetic $\iff$ $\omega_k$'s defined $\overline{\partial}$. (This definition is only possible because, as a CM elliptic curve, $\mathfrak{c}$ not only has 1st cohomology reps but is defined $\overline{\partial}$.) Then $(\mathfrak{c})$ is spanned by arithmetic classes.

Working with P. Griffiths & M. Green, we have defined a way to evaluate classes in $(\mathfrak{c})$ at points in

$$U = \left\{ (p, \ell), (p', \ell') \in D_m \times D_n \mid p \neq p', \quad \ell \cap \ell' = \emptyset, \quad p, p' \in B \right\},$$

and there is a notion of compatible pairs of CM points in $U$ such that the following holds:

Theorem (G & K): Arithmetic cuspidal automorphic cohomology classes take algebraic values on compatible CM pairs.

A general "philosophical" interpretation of this, seems a ways off.