1.9.6

\[ E\left( \frac{X-M}{\sigma} \right) = \frac{E(X) - M}{\sigma} = \frac{M - M}{\sigma} = 0 \]

\[ E\left[ \left( \frac{X-M}{\sigma} \right)^2 \right] = \frac{1}{\sigma^2} E\left[ (X-M)^2 \right] = \frac{1}{\sigma^2} \cdot \sigma^2 = 1 \]

\[ E\{ \exp[t\left( \frac{X-M}{\sigma} \right)] \} = E\{ e^{\frac{tx}{\sigma}} \cdot e^{-\frac{tm}{\sigma}} \} = e^{-\frac{tm}{\sigma}} E\left( e^{\frac{tx}{\sigma}} \right) = e^{-\frac{tm}{\sigma}} M\left( \frac{t}{\sigma} \right) \]

1.9.19

Expanding \( M(t) = (1-t)^{-3} \) at 0, we have

\[
(1-t)^{-3} = 1 + \frac{3!}{2} t + \frac{4!}{2} \frac{t^2}{2!} + \cdots + \frac{(k+2)!}{2} \frac{t^k}{k!} + \cdots
\]

Therefore,

\[ M'(0) = E(X) = \frac{3!}{2} = 3 \quad , \quad M''(0) = E(X^2) = \frac{4!}{2} = 12 \quad \cdots \]

In general,

\[ E(X^k) = \frac{(k+2)!}{2} \quad , \quad \text{for} \quad k = 1, 2, 3, \ldots \]
\[ M(t) = E(e^{tx}) = \int_{0}^{\infty} e^{tx} f(x) \, dx \]
\[ = \int_{0}^{\infty} e^{tx} \frac{1}{\beta} e^{-\frac{x}{\beta}} \, dx \]
\[ = \frac{1}{\beta} \int_{0}^{\infty} e^{(t-\frac{1}{\beta})x} \, dx \]
\[ = \frac{1}{\beta} \cdot \frac{1}{t-\frac{1}{\beta}} e^{(t-\frac{1}{\beta})x} \bigg|_{0}^{\infty} \]
\[ = \frac{1}{1-\beta t}, \quad t < \frac{1}{\beta}. \]

\[ E(X) = M'(0) = \frac{B}{(1-\beta t)^2} \bigg|_{t=0}^{t=0} = \beta \]
\[ E(X^2) = M''(0) = \frac{2\beta^2}{(1-\beta t)^2} \bigg|_{t=0}^{t=0} = 2\beta^2 \]
\[ \text{Var}(X) = E(X^2) - [E(X)]^2 = 2\beta^2 - \beta^2 = \beta^2 \]
First, we prove results in 1.10.4 (students still get credits without this step).

For \(0 < t < h\), \(X > a\) is equivalent to \(e^{tx} > e^{ta}\).

View \(e^{tx}\) be a random variable.

Let \(u(x) = e^{tx}\), apply Markov's inequality in Theorem 1.10.2, we have at \(c = e^{ta}\):

\[
P(X \geq a) = P(u(X) \geq c) \leq \frac{E(u(X))}{c} = \frac{E(e^{tx})}{e^{ta}} = e^{-at}M(t).
\]

Similar proof applies to the case \(-h < t < 0\).

Next, as the above inequality holds for all \(0 < t < h\),

\[
P(X \geq 1) \leq e^{-t}M(t) = \frac{1 - e^{-2t}}{2t},
\]

the right-hand side is continuous in \(t\), let \(t \to \infty\), we get

\[
P(X \geq 1) \leq \lim_{t \to \infty} \frac{1 - e^{-2t}}{2t} = 0.
\]

Meanwhile, \(P(X \geq 1) \geq 0\) as a probability. So, \(P(X \geq 1) = 0\).

The same argument applies to \(P(X \leq -1) = 0\).
1.10.6 All questions are applications of Jensen’s inequality. So, we just need to establish the convexity of the corresponding function in each case. Note that, we only need to focus on $x > 0$, as $P(x \leq 0) = 0$.

(a) $(\frac{1}{x})'' = (-\frac{1}{x^2})' = \frac{2}{x^3} > 0$.

(b) $(-\log x)' = (-\frac{1}{x})' = \frac{1}{x^2} > 0$

(c) $\log(\frac{1}{x}) = -\log x$, so it holds from (b).

(d) $(x^3)'' = (3x^2)' = 6x > 0$. 