Let $p_{00}, p_{11}, p_{10}, p_{01}$ denote the joint pmf at $(x, y) = (0,0), (1,1), (1,0), (0,1)$, respectively. Then

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$$

$$= (p_{11} - p_{01}) - (1 - p_{00}) (p_{11} - p_{01})$$

(Note: $p_{00} + p_{11} + p_{10} + p_{01} = 1$)

$$= p_{00} (p_{11} - p_{01})$$

Therefore, $X$ and $Y$ are uncorrelated if $p_{11} = p_{01}$.

On the other hand, if $X$ and $Y$ are independent, we need to have, for example,

$$P(X=0, Y=0) = P(X=0)P(Y=0),$$

which requires

$$p_{00} = p_{00} (p_{00} + p_{10}) \iff p_{00} + p_{10} = 1.$$ 

This cannot be true because all four probabilities are required to be positive. So, $X$ and $Y$ are not independent.

2.6.1 (a) Due to symmetry, the marginal pdf of $X$, $Y$, and $Z$ have the same form

$$f_X(x) = \int_0^1 \int_0^1 f(x,y,z) \, dy \, dz$$

$$= \int_0^1 \frac{2}{3} (x+z+\frac{1}{2}) \, dz$$

$$= \frac{2}{3} (x+1), \quad 0 < x < 1.$$

(b) $P(0 < x < \frac{1}{2}, 0 < y < \frac{1}{2}, 0 < z < \frac{1}{2}) = \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \frac{2}{3} (x+y+z) \, dx \, dy \, dz = \frac{2}{3} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} (\frac{y}{2} + \frac{x}{2}) \, dy \, dz = \frac{1}{3} \int_0^{\frac{1}{2}} (\frac{z}{2} + \frac{1}{4}) \, dz = \frac{1}{16}$

$P(0 < x < \frac{1}{2}) = \int_0^{\frac{1}{2}} f_X(x) \, dx = \frac{5}{12}$
(c) Since
\[ P(0 < x < \frac{1}{2}, 0 < y < \frac{1}{2}, 0 < z < \frac{1}{2}) \neq P(0 < x < \frac{1}{2}) \cdot P(0 < y < \frac{1}{2}) \cdot P(0 < z < \frac{1}{2}) \]

\( x, y \) and \( z \) are not independent.

(d) \[ E(XY \, Z) = \int_0^1 \int_0^1 \int_0^1 x y \, z \, \frac{2}{3} (x + y + z) \, dx \, dy \, dz \]
\[ = \frac{2}{3} \int_0^1 \int_0^1 x y (\frac{x + y}{2} + \frac{1}{3}) \, dx \, dy \]
\[ = \frac{2}{3} \int_0^1 x^2 (\frac{x}{4} + \frac{1}{3}) \, dx \]
\[ = \frac{25}{216} \]

\[ E(3 XY^2 Z^2) = \int_0^1 \int_0^1 \int_0^1 x y^2 \, z^2 \, \frac{2}{3} (x + y + z) \, dx \, dy \, dz \]
\[ = \frac{2}{3} \int_0^1 \int_0^1 y^2 \, z^2 (\frac{1}{3} + \frac{y + z}{2}) \, dy \, dz \]
\[ = \frac{2}{3} \int_0^1 y^4 (\frac{y}{6} + \frac{17}{72}) \, dy \]
\[ = \frac{1}{20} \]

\[ E(X^2 Y Z + 3 X Y^2 Z^2) = \frac{25}{216} + 3 \cdot \frac{1}{20} = \frac{28.7}{108} = 0.2657 \]

(e) Again, due to symmetry, the cdfs of \( x, y, z \) are of the same form.

\[ F_X(x) = \int_0^x f_x(t) \, dt = \int_0^x \frac{2}{3} (t + 1) \, dt = \frac{2}{3} \left( \frac{x^2}{2} + x \right), \]

\[ F_X(x) = 1 \quad \text{for} \ x > 1, \]

\[ F_X(x) = 0 \quad \text{for} \ x < 0. \]
The conditional pdf of $(X,Y) | Z = z$ is

\[
(f) \quad f_{X \mid Y \mid Z}(x, y \mid z) = \frac{f(x, y, z)}{f_z(z)}
\]

\[
= \frac{\frac{2}{3} (x+y+z)}{\frac{2}{3} (z+1)} = \frac{x+y+z}{z+1}, \quad 0 < x < 1, \quad 0 < y < 1.
\]

\[
E(X + Y \mid z) = \int_0^1 \int_0^1 (x+y) \frac{x+y+z}{z+1} \, dx \, dy
\]

\[
= \frac{1}{(z+1)} \int_0^1 (y^2 + (z+1)y + \frac{1}{3} + \frac{z}{2}) \, dy
\]

\[
= 1 + \frac{1}{6(z+1)}
\]

(g) The joint marginal pdf of $(Y, Z)$ is

\[
f_{Y, Z}(y, z) = \int_0^1 f(x, y, z) \, dx = \frac{2}{3} (y + z + \frac{1}{2}), \quad 0 < y < 1, \quad 0 < z < 1.
\]

Then $X \mid Y, Z$ has pdf

\[
f_{X \mid Y, Z}(x \mid y, z) = \frac{f(x, y, z)}{f_{Y, Z}(y, z)} = \frac{x+y+z}{y+Z+\frac{1}{2}}, \quad 0 < x < 1.
\]

\[
E(X \mid Y, Z) = \int_0^1 x \frac{x+y+z}{y+Z+\frac{1}{2}} \, dx = \frac{3(y+z)+2}{6(y+z)+3}
\]
2.6.6 \[ \text{Cov}(X_1, X_2) = E[(X_1 - \mu_1)(X_2 - \mu_2)] \]
\[ = E_{(X_2, X_3)}\left[ E((X_1 - \mu_1)(X_2 - \mu_2) | X_2, X_3) \right] \]
\[ = E_{(X_2, X_3)}\left[ (X_2 - \mu_2) E(X_1 - \mu_1 | X_2, X_3) \right] \]
\[ = E_{(X_2, X_3)}\left[ b_2 (X_2 - \mu_2)^2 + b_3 (X_2 - \mu_2)(X_3 - \mu_3) \right] \]
\[ = b_2 \text{Var}(X_2) + b_3 b_2 \text{Cov}(X_2, X_3) \]

That is, \( \rho_{12} \sigma_1 \sigma_2 = b_2 \sigma_2^2 + b_3 \rho_{23} \sigma_2 \sigma_3 \) \hspace{1cm} (1)

Similarly, looking at \( \text{Cov}(X_1, X_3) \) gives
\( \rho_{13} \sigma_1 \sigma_3 = b_2 \rho_{23} \sigma_2 \sigma_3 + b_3 \sigma_3^2 \)
\hspace{1cm} (2)

Solving (1) and (2), we have
\[ b_2 = \frac{\sigma_1 (\rho_{12} - \rho_{23})}{\sigma_2 (1 - \rho_{23}^2)} \]
\[ b_3 = \frac{\sigma_1 (\rho_{13} - \rho_{12} \rho_{23})}{\sigma_3 (1 - \rho_{23}^2)} \]

2.7.7 \( X_1 = Y_1 Y_2 Y_3 Y_4, X_2 = Y_2 Y_3 Y_4, X_3 = Y_3 Y_4, X_4 = Y_4 \)

It is easy to see that \((X_1, X_2, X_3, X_4) \leftrightarrow (Y_1, Y_2, Y_3, Y_4)\) is one-to-one

and since \(0 < X_1 < X_2 < X_3 < X_4 < 1\), we have \(0 < Y_1, Y_2, Y_3, Y_4 < 1\),
which leads to \(Y_1 Y_2 Y_3 Y_4 < 1\).

The Jacobian is
\[ J = \begin{vmatrix}
    y_1 y_2 y_3 y_4 & y_1 y_3 y_4 & y_1 y_2 y_4 & y_1 y_2 y_3 \\
    0 & y_3 y_4 & y_2 y_4 & y_2 y_3 \\
    0 & 0 & y_4 & y_3 \\
    0 & 0 & 0 & 1
\end{vmatrix} = y_2 y_3 y_4^3 \]

Therefore, using the change-of-variable formula,
\[ f(y_1, y_2, y_3, y_4) = f(x_1, x_2, x_3, x_4) \cdot |J| = 24 y_2 y_3^2 y_4^3, \]
\(0 < y_1, y_2, y_3, y_4 < 1\).

Since \(f(y_1, y_2, y_3, y_4)\) can be factorized, by Thm 2.5.1 (extension to the multivariate case), \(Y_1, Y_2, Y_3, Y_4\) are independent.
2.8.14 Note $\text{Cov}(x_1, x_2) = \rho \sigma_1 \sigma_2$

\[
\text{Cov}(\gamma, \zeta) = \text{Cov}(a_1 x_1 + a_2 x_2, b_1 x_1 + b_2 x_2)
\]
\[
= \text{Cov}(a_1 x_1, b_1 x_1) + \text{Cov}(a_1 x_1, b_2 x_2) + \text{Cov}(a_2 x_2, b_1 x_1) + \text{Cov}(a_2 x_2, b_2 x_2)
\]
\[
= a_1 b_1 \sigma_1^2 + a_1 b_2 \rho \sigma_1 \sigma_2 + a_2 b_1 \rho \sigma_1 \sigma_2 + a_2 b_2 \sigma_2^2
\]

\[
\text{Cor}(\gamma, \zeta) = \frac{\text{Cov}(\gamma, \zeta)}{\sqrt{\text{Var}(\gamma)} \sqrt{\text{Var}(\zeta)}} = \frac{a_1 b_1 \sigma_1^2 + (a_1 b_2 + a_2 b_1) \rho \sigma_1 \sigma_2 + a_2 b_2 \sigma_2^2}{\sqrt{a_1^2 \sigma_1^2 + 2 a_1 a_2 \rho \sigma_1 \sigma_2 + a_2^2 \sigma_2^2 / b_1^2 \sigma_1^2 + 2 b_1 b_2 \rho \sigma_1 \sigma_2 + b_2^2 \sigma_2^2}}
\]