Homework 9

3.3.6 $F_Y(y) = P(Y \leq y) = 1 - P(Y > y)$

$= 1 - P(X_1 > y, X_2 > y, X_3 > y)$

$= 1 - \left( \int_{y}^{\infty} e^{-x} \, dx \right)^3$

$= 1 - e^{-3y}$, for $0 < y < \infty$

Then $f_Y(y) = F'_Y(y) = 3e^{-3y}$, $0 < y < \infty$

And this is the exponential distribution with parameter $\lambda = 3$.

(b) $F_Y(y) = P(Y \leq y) = P(X_1 \leq y, X_2 \leq y, X_3 \leq y)$

$= \left( \int_{0}^{y} e^{-x} \, dx \right)^3$

$= \left( 1 - e^{-y} \right)^3$, for $0 < y < \infty$

$f_Y(y) = 3 e^{-y} \left( 1 - e^{-y} \right)^2$, $0 < y < \infty$.

3.3.15 In general

$P(X=x)$: A more rigorous way to state the given conditions is

$X | M=m \sim \text{Poisson}(m)$ and $M \sim \text{Gamma}(\alpha=2, \beta=1)$.

Following the law of total probability,

$P(X=x) = \int_{0}^{\infty} P(X=x | M=m) f(m) \, dm$

$= \int_{0}^{\infty} \frac{m^x e^{-m}}{x!} \cdot \frac{1}{\Gamma(2)} \cdot m^{2-1} e^{-m/1} \, dm$

$= \frac{1}{\Gamma(2)} \cdot \frac{1}{1} \cdot \int_{0}^{\infty} \frac{m^x e^{-m}}{x!} \, dm$
\[ = \int_0^\infty \frac{m^{x+1} e^{-2m}}{x!} \, dm \quad \text{(Note } \Gamma(x) = 1, \Gamma(1) = 1, x = 1, \text{)} \]

\[ = \frac{\Gamma(x+2)}{x!} \left( \frac{1}{2} \right)^{x+2} \int_0^\infty \frac{1}{\Gamma(x+2)} \left( \frac{1}{2} \right)^{x+2} m^{(x+2)-1} e^{-m/2} \, dm \]

\[ = \frac{(x+1)!}{x!} \frac{1}{2^{x+2}} \quad \text{(The above integrand is the pdf of gamma)} \]

\[ = \frac{x+1}{2^{x+2}} \quad \text{(} \Gamma(x+2) = (x+1)! \text{ for integer } x \geq 0, 1, 2, \ldots \text{)} \]

Then \[ P(X=0) = \frac{1}{4}, \quad P(X=1) = \frac{1}{4}, \quad P(X=2) = \frac{3}{16}. \]

\[ \text{Note} \] that, since \( M \) is a continuous random variable, it is not straightforward to use the law of total probability and we used

an analogy. The rigorous justification requires higher level math beyond this class.

**3.3.16** \( F_Y(y) = P(Y \leq y) = P(-2 \log X \leq y) \)

\[ = P(X \geq e^{-2y}) \]

\[ = \int_1^{\infty} 1 \cdot dx \]

\[ = 1 - e^{-2y}, \quad \text{for } 0 < y < \infty \]

And the pdf is \( f_Y(y) = F'_Y(y) = 2e^{-2y}, \quad 0 < y < \infty \).

\[ \text{Note this is exponential distribution with parameter } \lambda = 2. \]

**Remark:** This transformation how one may obtain random numbers from an exponential distribution based on random numbers from \( U(0,1) \).
3.3.24 (a) 

\[ M_Y(t) = E(e^{tX}) = E(e^{2tx_1 + 6tx_2}) \]
\[ = E(e^{2tx_1}) \cdot E(e^{6tx_2}) \quad \text{(due to independence)} \]
\[ = M_1(2t) \cdot M_2(6t) \]
\[ = \frac{1}{(1 - 3\cdot 2t)^3} \cdot \frac{1}{(1 - 1.6t)^5} \]
\[ = \frac{1}{(1 - 6t)^8}, \quad \text{for } t < \frac{1}{6}. \]

(b) The form of the MGF of \( Y \) suggests that
\[ Y \sim \text{gamma}(\alpha = 8, \beta = 6). \]

3.3.27 (a) According to the setup in Example 3.3.7, Using Theorem 3.3.2, we have \( X_2 + \cdots + X_{k+1} \sim \text{gamma}(\sum_{i=2}^{k+1} \alpha_i, \beta = 1). \) And \( X_i \) is independent of \( X_2 + \cdots + X_{k+1} \), then using the constructive definition of a beta random variable based on two gamma random variables (details in pages 162-163), we know \( Y_i = \frac{X_i}{X_1 + (X_2 + \cdots + X_{k+1})} \sim \text{beta}(\alpha_i, \sum_{i=2}^{k+1} \alpha_i) \)

For \( r \leq k \),
\[ Y_1 + \cdots + Y_r = \frac{X_1 + \cdots + X_r}{X_1 + \cdots + X_{k+1}} = \frac{X_1 + \cdots + X_r}{(X_1 + \cdots + X_r) + (X_{k+1} + \cdots + X_{k+1})} \]

from Thm 3.3.2

\[ X_1 + \cdots + X_r \sim \text{gamma}(\sum_{i=1}^{r} \alpha_i, \beta = 1) \] and \( X_{k+1} + \cdots + X_{k+1} \sim \text{gamma}(\sum_{i=k+1}^{k+1} \alpha_i, \beta = 1) \)

And since the two parts are independent, we have
\[ Y_1 + \cdots + Y_r \sim \text{beta}(\sum_{i=1}^{r} \alpha_i, \sum_{i=k+1}^{k+1} \alpha_i) \]
(c) Let 
\[ \tilde{X}_1 = (X_1 + X_2) \sim \text{gamma}(\alpha_1 + \alpha_2, \beta = 1) \]
\[ \tilde{X}_2 = (X_3 + X_4) \sim \text{gamma}(\alpha_3 + \alpha_4, \beta = 1) \]
\[ \tilde{X}_3 = X_5 \sim \text{gamma}(\alpha_5, \beta = 1) \]
\[ \vdots \]
\[ \tilde{X}_k = X_{k+2} \sim \text{gamma}(\alpha_{k+2}, \beta = 1) \]
\[ \vdots \]
\[ \tilde{X}_{k-1} = X_{k+1} \sim \text{gamma}(\alpha_{k+1}, \beta = 1) \]

Then
\[ Y_1 + Y_2 = \frac{X_1 + X_2}{X_1 + \cdots + X_{k+1}} = \frac{\tilde{X}_1}{\tilde{X}_1 + \tilde{X}_2 + \cdots + \tilde{X}_{k-1}} \]
\[ Y_3 + Y_4 = \frac{\tilde{X}_2}{\tilde{X}_1 + \tilde{X}_2 + \cdots + \tilde{X}_{k-1}} \]
\[ \vdots \]
\[ Y_{i+2} = \frac{\tilde{X}_i}{\tilde{X}_1 + \cdots + \tilde{X}_{k-1}} \]
\[ \vdots \]
\[ Y_k = \frac{\tilde{X}_{k-2}}{\tilde{X}_1 + \cdots + \tilde{X}_{k-1}} \]

Since all \( \tilde{X}_i \)'s are independent gamma random variables with the same value \( \beta = 1 \), from the result in Example 3.3.7, we know \( (Y_1 + Y_2), (Y_3 + Y_4), \ldots, Y_k \) have a Dirichlet distribution with parameters \( \alpha_1 + \alpha_2, \alpha_3 + \alpha_4, \alpha_5, \ldots, \alpha_k, \alpha_{k+1} \).