Name: ________________________________

Instructions: You have 2 hours to take this exam. You may use one 3 x 5 notecard. Calculators are not allowed.

Show your work and simplify your answers.
1. (a) Find the maximum and minimum values of the function \( f(x, y) = x^2 + 3y^2 \) subject to the constraints \( x + y = 1 \), \( x \geq 0 \), \( y \geq 0 \).

\[
\begin{align*}
x &= 1 - y, \quad 0 \leq y \leq 1 \\
\frac{\partial f}{\partial y} &= (1-y)^2 + 3y^2 = 1 - 2y + 4y^2 \\
\frac{\partial^2 f}{\partial y^2} &= -2 + 8y = 0 \\
y &= \frac{1}{4}, \quad x = 1 - y = \frac{3}{4} \\
\text{2nd deriv. test: } f''(y) &= -2 < 0 \rightarrow \text{ local max}
\end{align*}
\]

\[
\begin{align*}
\text{minimum occurs at } \quad x &= \frac{3}{4}, \quad y = \frac{1}{4} \\
\text{minimum value: } f\left(\frac{3}{4}, \frac{1}{4}\right) &= \frac{9}{16} + \frac{1}{16} = \frac{10}{16} = \frac{5}{8}
\end{align*}
\]

Check endpoints for max/min:

\[
\begin{align*}
y &= 0 \quad f(y) = 1 \\
y &= 1 \quad f(y) = 1 - 2(1) = 3 \\
\rightarrow \text{max value is } 3
\end{align*}
\]

(b) Find the maximum area of a rectangle that can be inscribed between the curve \( y = 1 - x^2 \) and the x-axis, between \( x = -1 \) and \( x = 1 \).

\[
\begin{align*}
y &= 1 - x^2 \\
A &= 2xy = 2x(1-x^2) = 2x - 2x^3 \\
A' &= 2 - 6x^2 = 0 \\
x &= \frac{1}{\sqrt{3}} \\
x &= \frac{\sqrt{3}}{3} \quad \text{(discard negative root)} \\
y &= 1 - \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{2}{3} \\
A &= 2xy = 2 \left(\frac{1}{\sqrt{3}}\right) \left(\frac{2}{3}\right) = \frac{4\sqrt{3}}{9} \\
&\approx \frac{1.35}{4}
\end{align*}
\]
2. The volume of a cylinder of height $h$ and base radius $r$ is $\pi r^2 h$, and it surface area is $2\pi r^2 + 2\pi rh$. What are the dimensions of a cylinder of the largest possible volume with surface area $10\pi \text{ cm}^2$?

\[
\begin{align*}
\text{constraint} & \quad 2\pi r^2 + 2\pi rh = 10\pi \\
\pi r^2 + rh &= 5 \\
h &= \frac{5 - r^2}{r}
\end{align*}
\]

\[
V = \pi r^2 h = \pi r^2 \left( \frac{5 - r^2}{r} \right) = \pi r \left( 5 - r^2 \right) = 5\pi r - \pi r^3
\]

\[
V' = 5\pi - 3\pi r^2 = 0
\]

\[
3r^2 = 5 \\
r = \sqrt[3]{\frac{5}{3}} \quad \text{(discard negative root)}
\]

\[
h = \frac{5 - \sqrt[3]{\frac{5}{3}}}{\sqrt[3]{\frac{5}{3}}} = \frac{15/3 - \sqrt[3]{5/3}}{\sqrt[3]{5/3}} = \frac{10/3}{\sqrt[3]{5/3}} = 2 - \frac{5}{3} \cdot \sqrt[3]{\frac{5}{3}}
\]

\[
= 2 \sqrt[3]{\frac{5}{3}}
\]
3. Suppose the demand function for some product is determined to be \( p(x) = 40 - 0.02x \), and the cost function is \( C(x) = 8000 + 4x \).

(a) At what price should the product be sold to maximize revenue, and what would the revenue be in this case?

\[
R(x) = x(40 - 0.02x) = 40x - 0.02x^2
\]

\[
R'(x) = 40 - 0.04x = 0
\]

\[x = \frac{40}{0.04} = 1000\]

\[p(x) = 40 - 0.02(1000) = 40 - 20 = \boxed{20}\] price

\[R(1000) = 1000(20) = \boxed{20,000}\] revenue

(b) At what price should the product be sold to maximize profit, and how many units would be sold in this case?

\[
P(x) = R(x) - C(x) = (40x - 0.02x^2) - (8000 + 4x)
\]

\[= -0.02x^2 + 36x - 8000
\]

\[P'(x) = -0.04x + 36 = 0
\]

\[x = \frac{36}{0.04} = 900\] units

\[p = 40 - 0.02(900) = 40 - 18 = \boxed{22}\] price
4. Simplify, if possible.

(a) $\ln\left(\frac{1}{\sqrt{e}}\right) = \ln\left(e^{-\frac{1}{2}}\right) = -\frac{1}{2}$

(b) $\ln(1) = 0$

(c) $\ln(x - y) \quad \text{[doesn't simplify]}

(d) $\ln(x^3) - \ln(x^2) = 3\ln x - 2\ln x = \ln x$

(e) $\ln x - \ln(1/x) = \ln x - \ln x^{-1} = \ln x + \ln x = 2\ln x$

(\because \ln x^2)$
5. Solve for $x$, if possible.

(a) $2e^{3x} = 4$

\[ e^{3x} = 2 \]

\[ 3x = \ln 2 \]

\[ x = \frac{1}{3} \ln 2 \]

(b) $5^x = 50$

\[ x \cdot \ln 5 = \ln 50 \]

\[ x = \frac{\ln 50}{\ln 5} \]

(c) $\ln(3x + 1) = 2$

\[ 3x + 1 = e^2 \]

\[ x = \frac{e^2 - 1}{3} \]

(d) $\ln(x + 1) - \ln x = 1$

\[ \frac{\ln(x + 1)}{\ln x} = 1 \]

\[ \ln x = e \]

\[ x + 1 = e \cdot x \]

\[ x = \frac{1}{e - 1} \]

(e) $3e^{7x} = 5e^{5x}$

\[ e^{2x} = \frac{5}{3} \]

\[ 2x = \ln \left( \frac{5}{3} \right) \]

\[ x = \frac{1}{2} \ln \left( \frac{5}{3} \right) \]
6. A bacteria colony grows exponentially. At 12:00 there are 4,000 bacteria and at 3:00 there are 6,000 bacteria.

(a) Write a formula for the number of bacteria \( t \) hours after 12:00.

\[
\rho = A e^{kt}
\]
\[
A = 4000
\]
\[
6000 = 4000 e^{3h}
\]
\[
e^{\frac{3h}{2}} = 9
\]
\[
3h = \ln(9) = \ln(3^2)
\]
\[
k = \frac{1}{3} \ln(3)
\]

\[
\rho = 4000 e^{\frac{1}{3} \ln(3) t}
\]

(b) At what time will there be 9,000 bacteria?

\[
4000 e^{\frac{1}{3} \ln(3) t} = 9000
\]
\[
\frac{1}{3} \ln(3) t = \ln(\frac{9}{4})
\]
\[
\ln(\frac{9}{4}) = \frac{3 \ln(3)}{3} = \frac{2 \ln(3)}{3} = 6 \text{ hours after 12:00}
\]

(Alt. solution: 12:00 \( \rightarrow \) 3:00 population goes from 4,000 to 6,000 = 50% increase, so 3:00 \( \rightarrow \) 6:00 will go from 6,000 to 9,000 = another 50% increase, driven by 50% every 3 hours.)
7. (a) $1,000 is invested in an account that pays 4% interest. How long must we wait to get $1,500, if the interest is compounded (a) yearly; (b) monthly; (c) continuously?

(a) \[ 1000 (1 + 0.04)^t = 1500 \]
   \[ (1 + 0.04)^t = \frac{1500}{1000} \]
   \[ t \ln (1.04) = \ln 1.5 \]
   \[ t = \frac{\ln 1.5}{\ln 1.04} \]

(b) \[ 1000 (1 + \frac{0.04}{12})^{12t} = 1500 \]
   \[ (1 + \frac{0.04}{12})^{12t} = 1.5 \]
   \[ 12t \ln (1 + \frac{0.04}{12}) = \ln 1.5 \]
   \[ t = \frac{\ln 1.5}{12 \ln (1 + \frac{0.04}{12})} \]

(c) \[ 1000 e^{0.04t} = 1500 \]
   \[ e^{0.04t} = 1.5 \]
   \[ 0.04t = \ln 1.5 \]
   \[ t = \frac{\ln 1.5}{0.04} \]

(b) How much money must be invested now to get $1,500 in 10 years, if the account yields 4% interest compounded (a) yearly; (b) monthly; (c) continuously?

(a) \[ P (1 + 0.04)^{10} = 1500 \]
   \[ P = \frac{1500}{(1 + 0.04)^{10}} \]

(b) \[ P (1 + \frac{0.04}{12})^{120} = 1500 \]
   \[ P = \frac{1500}{(1 + \frac{0.04}{12})^{120}} \]

(c) \[ P e^{-0.4} = 1500 \]
   \[ P = \frac{1500}{e^{-0.4}} \]
8. Find the derivatives of the following functions.

(a) $3e^{4x}$

$$12e^{4x}$$

(b) $3e^x$

$$3e^x$$

(c) $xe^{x^2}$

$$e^x + 2xe^{x^2}$$

(d) $e^x\sin x$

$$\left(e^x \sin x + x e^x \cos x\right) e^x \sin x$$

(e) $\frac{e^x + e^{-x}}{2}$

$$\frac{e^x - e^{-x}}{2}$$
9. Find the derivative of the function

\[ f(x) = e^{2x} \tan(3x^2 - 1) \sin(e^{5x}). \]

\[
\left[ 5x^4 \tan(3x^2 - 1) + 6x^6 \sec^2(3x^2 - 1) \right] e^{5x} \tan(3x^2 - 1) \sin(e^{5x}) \\
+ 5e^{5x} + x^5 \tan(3x^2 - 1) \cos(e^{5x})
\]
10. (a) Find the intervals on which the function \( y = e^{-x^2} \) is increasing and decreasing.

\[
\text{increasing: } -2x e^{-x^2} > 0 \\
\text{always positive}
\]

\[-2x > 0 \]

\[x < 0 \]

\[
\text{decreasing when } x > 0
\]

(b) Find the intervals on which the function \( y = e^{-x^2} \) is concave up and concave down.

\[
\text{concave up: } -2e^{-x^2} + 4x^2 e^{-x^2} > 0
\]

\[
(-2 + 4x^2) e^{-x^2} > 0
\]

\[
\text{always positive}
\]

\[-2 + 4x^2 > 0 \]

\[x^2 > \frac{1}{2} \]

\[x > \sqrt{\frac{1}{2}} \text{ or } x < -\sqrt{\frac{1}{2}} \]

\[
\text{concave up when } x < -\sqrt{\frac{1}{2}} \text{ or } x > \sqrt{\frac{1}{2}}
\]

\[
\text{concave down when } -\sqrt{\frac{1}{2}} < x < \sqrt{\frac{1}{2}}
\]